The Pythagorean Theorem as a Rooted In-tree Dependency Graph

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Abstract

We look back to concept and dependency graphs of Euclid's *Elements,* Book 1, that show the deductive relationships among its propositions. We claim that Book 1 does not have one overall coherent structure but rather is organized around two propositions—1.45 and 1.47. In other words, Book 1 has a dual core. For the latter proposition, the Pythagorean Theorem, we constructed a rooted in-tree graph to help visualize its role and proof.

Background

Computer programs such as R [1] provide new tools for analyzing proofs and dependencies within the deductive system of Euclid's *Elements,* Book 1 [2]. However, the results of our hand-generated dependency graphs, both 2-D and 3-D, lead the way to one answer to a perennial question. What role does the Pythagorean Theorem play in the overall design of Book 1? Did Euclid consider 1.47 as that book's core around which the whole book is organized or are there other propositions within Book 1, such as 1.45, that serve a similar role or even better fit this role? Perhaps there is no core at all. Propositions 1.45 and 1.47 read as follows:

1.45 *To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.* [2]

1.47 *In right-angled triangles the side subtending the right angle is equal to the squares on the sides containing the right angle.* [2]

The Last 150 Years

Eugene Boman [3] in his tantalizing paper, *Euclid21,* summarizes his group's recent work on Book 1 and, as well, documents previous attempts to graph that book's entire structure. As early as 1879, Boman notes, Charles Dodgson (Lewis Carroll), in his own book on Euclid [4], made a partial sketch of the relations among Book 1's propositions. Brett Eyer, Boman's student, completed an earlier draft of his teacher's partial sketch of Book 1 (fig.1) that outlines the proof of the Pythagorean Theorem. They conclude that their figure "clearly refutes the idea that Book 1 is organized around the Pythagorean Theorem." This conclusion follows, they say, because, while the shaded propositions support the Pythagorean Theorem, there are seven propositions in the book that do not [more like 15]. "The Pythagorean Theorem is clearly *a* focal point of Book 1, but it is not *the* focal point" [3]. By "the focal point" or "core," we assume they mean a proposition that, if missing from the book, would eviscerate that book's overall logical structure. However, since there is no proposition in Book 1 as it now stands that uses all of the propositions of that book in its proof, it follows from Eyer's criterion that there is no "the-core" possible. However, this still leaves room for "a-core," or even more than one, cores that do not use all of the Book's propositions but still are essential to its integrity.
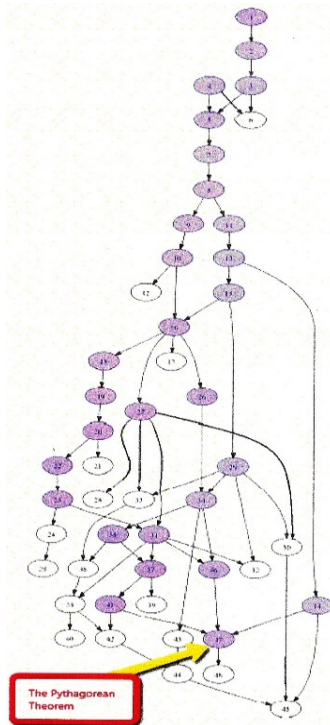

Figure 1 *Eyer's Proof of the 1.47*

There have been other graphs, one in Mark Schiefsky's in-depth look at *Elements* [5]. He calls his a "dependency" graph for Book 1 (fig. 2). Schiefsky wonders if 1.45 (at bottom center in his figure) might
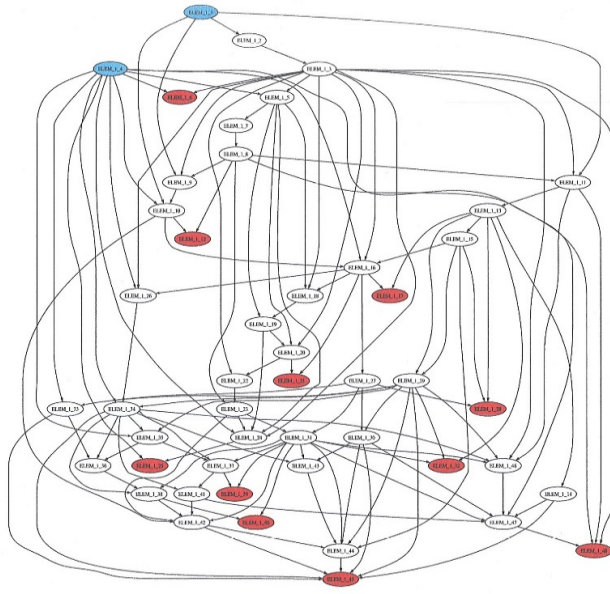
Figure 2 *Schiefsky's Dependency Graph, Book 1, 1.45, Bottom Center*

be the more dominate proposition in Book 1 because it uses the most propositions--two more than 1.47. He cites Mueller [6] who "claims that 1.45 is the most important consideration in determining the content and order of propositions in Book 1 in the sense that the analysis of the conditions of solution of the problem posed in 1.45 leads naturally back to the earlier propositions and theorems in the book." Hence, 1.45 is a plausible candidate for "a-core" in Book 1. This is not to say that Schiefsky or Mueller dismiss the importance of 1.47.

Boxer and Clutter [7], at the 2015 Bridge Conference, presented a succinct paper with a Book 1 concept (dependency) graph (not shown here but similar in appearance to that of Schiefsky, fig. 2). The software was R with an Rgraphviz package. After the fact (personal communication), they reluctantly concluded that there is no overall coherent structure to Book 1, though at the same time, their graph's value is that it highlights the two different kinds of propositions, "technological and logical," used in Book 1. Boxer suggested to us that our rooted in-tree graph (fig. 3) makes a case for a remaining core, namely the Pythagorean Theorem, though not a case for an overall coherent, or "grand plan."

No one of the graphs of Schiefsky, Boxer/Clutter, or Nyugen [8] (also not pictured here) seems particularly helpful in making it easy to visualize Book 1's logical dependencies. These graphs of all of Book 1 carry too much information. However, using the programs that generated them could yield through statistical analysis hidden relations among the dependencies—e.g., greatest, least, and cumulative distances among propositions, number of appearances, and mean, mode, and median distances (measured for each proof). As Schiefsky notes on page 26 [3], the latter indicate the contents of the toolbox used in any proof and should be explored.

Our Pythagorean Theorem Dependency Graph

Figure 3 shows our 3-D dependency graph (a piece of concept art) accepted into the 2016 Bridge's Art Show. While it provides no more information (truth) than our 2-D dependency graph (not pictured), we wanted to equate truth with beauty as well as simplify complexity. It includes as nodes or vertices (myrtle or pine wood balls, both woods of Ancient Greece), all 5 of Book 1's common notions, all 5 postulates, 3 of 23 definitions, and 31 of 48 propositions. At the top of the sculpture is proposition 1 (1.1). At the bottom center is proposition 47 (1.47). The connections between each node (1/8 inch brass rods) represent the in-tree directed dependencies, understood in this sense:

A depends upon B if B is necessary in the proof of A. (E.g., A needs B evaluated first.)

This relation isdefined in Nyugen's 2007 paper (with a stronger *iff*), where he presents dependency graphs of Book 1 similar to those of Boxer/Clutter and Schiefsky.

In our graph in fig. 3, the larger balls represent propositions in the proof. The smallest balls that often surround the larger balls are sometimes propositions but usually postulates, common notions (axioms), or definitions needed in that proposition's proof. Our graph is a directed rooted in-tree graph, meaning that all edges point towards the root (1.47) of the tree, except for those between definitions and propositions that do not involve a deduction. What makes the graph unique is that every proposition needed to prove

1.47 is itself "proven." If, however, a proposition appears more than once, then we cite it, that is, it becomes a node in the graph but is not proven again. This approach fills out the logical dependencies and allows analysis, e.g., of the number of times a proposition, definition, common notion, or postulate is a branch on the deductive tree. For example, 1.4 appears eight times in the tree, but we prove it only once. In our tree, we have branching that goes for 39 levels. Also, we can determine the distance any proposition is from the root and calculate accumulated distances, greatest distances, and least distances—all information about 1.47's toolbox contents. Much of this data we have summarized in Excel bar graphs (not pictured here).

Figure 3 *3-D Rendering of 2-D Rooted In-tree*
Graph of the Proof of the Pythagorean Theorem,
Euclid, Elements, Book 1

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Conclusion

The whole of the Pythagorean Theorem's proof is one of two cores of Book 1. Were we (or Boxer—personal communication) editing Euclid's original text, we would have suggested that 1.45, for all of the intellectual space it takes up in Book 1, be shifted into Book 2 where it is first needed. Its presence there would expand that book's meager content and 1.45's removal from Book 1 would lend more coherence to that book's system of proofs. Further, such adjustments (movements) are possible for other unused propositions and definitions. Overall, our preference would be for Book 1 to be exclusively 1.47.

However, Euclid clearly thought 1.45 important for Book 1. Likewise, so was 1.47, to say the least. Hence, we are satisfied that together these be the *dual core* of Book 1.

Acknowledgements

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