The Hendecagonal Stars in the Alhambra

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Abstract

The ceilings of the pavilions of the Court of Lions in the Alhambra (Granada, Spain) are masterpieces of the Nasrid woodwork. Made in pinewood by the 'ataurejado' technique they display a highly unusual decoration in *eleven-sided* or *hendecagonal* rosettes. The geometric explanation of the patterns given in the present paper considers the domes as a near miss polyhedron with eleven-sided faces or as a snub cube. It implies the cupolas of the pavilions can be considered as two early examples of (parts of) those polyhedra.

Introduction

The Alhambra of Granada in Spain is the most famous architectonic legacy of Nasrid dynasty (1238-1492), the last of the Muslim domination in 'Al-Andalus'. Built on a Roman villa, the complex includes military fortifications, gardens, fountains and several small yet pretty residences of the Nasrid sultans. The Nasrid architectural style coincides with the reign of Muhammed V (1354-1359, 1362–1391), when decorative elements inspired by nature were introduced. The epitome is the Court of Lions [7], a rectangular outdoor area, surrounded by a gallery of 120 columns in white marble, connected by '*sebka*' decorations of curvilinear arcs intersecting as rhombuses. The stone garden is the materialization of Paradise, and the svelte columns represent the palm trees of an oasis with in its center a fountain with twelve lions. At both ends of the court are pavilions with square bases and four-gabled roofs under both hemispheric cupolas with a diameter of 3.30 m. approximately, covered by ceilings with pinewood panels, dated from 1380 (Figure 1a and b). The panels follow the Nasrid style using the '*ataujerado*' technique of nailing the different pieces. The decorations display twelve big rosettes with eleven 'petals' ('*azafates*'), thus simulating the leafs of palm-trees (Figure 1c). Arte Historia shows a quality color picture [1], while the Victoria and Albert Museum of London exhibits a plaster model of one of the pavilions of the court with a very neat reconstruction of the spherical pattern of its ceiling [8].

Eleven-sided polygons were not frequent in Islamic art but they did occur, for instance in the interlocking nonagons, decagons and dodecagons on the decorations of domes or in a design on a Topkapı scroll, mixing hendecagons and nonagons [2]. There are also ancient planar patterns displaying heptagonal stars and/or rosettes together with other polygonal stars and/or rosettes, but patterns with *only* spatial hendecagonal main elements are highly unusual if not to say non-existing, at least in the data base and sources we consulted. Also, we are unaware of any recorded drawing explaining the design of the cupola decorations of the Court of Lions. Of course, the patterns have been drawn (Figure 2a) and described, in words, in works of scholars in art history [6], but the present paper provides the first geometric analysis of the decoration of the ceiling and a geometric explanation focusing on their spatial concept. The construction given here is not a spatial version of a flat pattern, but it proposes a design especially made for decorating a curved surface with planar components, based on an underlying polyhedral form.

Figure 1: *a) The western pavilion of the Court of Lions in Alhambra; b) the ceiling of the western cupola; c) Restoration of the eastern pavilion (photo P. Marín).*

The patterns of the two pavilions are similar, showing an equilateral (though not regular) octagon in the center, surrounded by twelve apparently regular hendecagonal rosettes. The lower part of the cupolas is decorated by narrow strips with geometric and floral elements having a merely ornamental function. The decorations of the central octagons in the two pavilions differ, so that they are usually distinguished as *western* or *eastern*. However, they are almost twins, and thus only one geometrical construction is necessary. We focus on the *eastern pavilion*, under restoration since October 2015 (Figure 1c).

The Patterns and Polyhedra

We first observe that the number of rosettes and their location correspond to those of a half of the *hendecagonal near miss polyhedron* (Figure 2b and [3]). This polyhedron can be obtained by a truncation of a pentagonal icositetrahedron. It provides a solid with 86 faces (24 hendecagons, 6 squares and 56 triangles of three types) none of them being regular polygons, but *very nearly regular*. The polyhedron can be adjusted so that only the hendecagons are irregular and there also exist spherical Islamic patterns derived from the near miss polyhedron [5], but none follows the design of the Court of Lions. Here we use another construction for approximating the solid, by grouping regular hendecagons edge to edge around a square, so that each hendecagon shares a vertex with the square, but again the fit will not be exact. The polyhedral model of the dome is designed supposing the hendecagonal faces are regular, then the "*regular*" 11-pointed stars have been drawn on computer, approximately, because an exact "ruler-compass" construction is of course impossible. However, the small irregularities in shape or in fit vanish in a practical realization (Figure 2c).

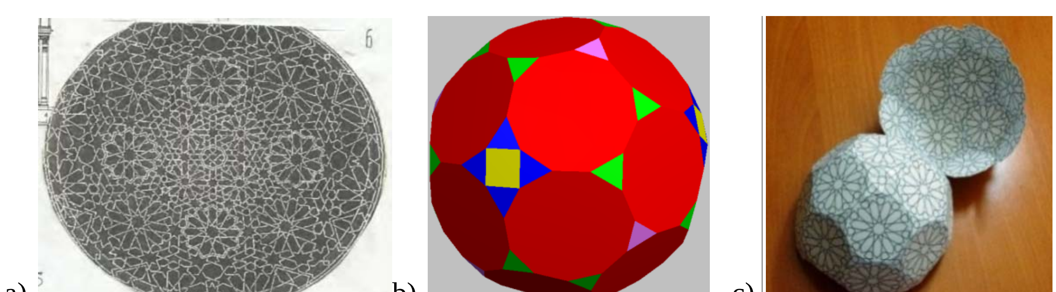


Figure 2: *a) a handmade drawing of the eastern pavilion (source: figure 60, 6 of [6]); b) hendecagonal near miss (using Stella4D); c) paper model of the dome from a near miss with regular hendecagons.*

The rosette (Figure 3a) displays a central (11/4.5)2 star (Figure 3b) surrounded by 11 concave quadrilaterals alternating with 11 irregular hexagons, which we will call ‘petals’. On their turn, they are surrounded by 11 irregular hexagons alternating with 11 concave octagons, which we will call ‘arrows’. Figure 3c shows the scheme of the construction, using the following steps:

1) Draw an auxiliary inner regular hendecagon.

2) Inscribe the (11/4.5)2 star; next, extend its sides to obtain 11 couples of half-lines *r*.

3) Extend the apothems of the precedent hendecagon to obtain 11 half-lines *s*.

4) Draw a circle passing through a point *P* and determine an imaginary hendecagon with vertices on the half-lines *s*. Next, inscribe the (11/2)2 star.

5) Draw a circle passing through a point *Q* and determine a hendecagon with vertices in the half-lines *s*, and inscribe the (11/3)1 star.

The rosette of real pattern shows five small eight pointed stars surrounding it (Figure 3a). These stars appear when the unit rosette is copied by reflection across a suitable line *t* perpendicular to the half-line *s*. (See [4] for information about Islamic stars and the notation *(n/d)s* for a real value *d*).

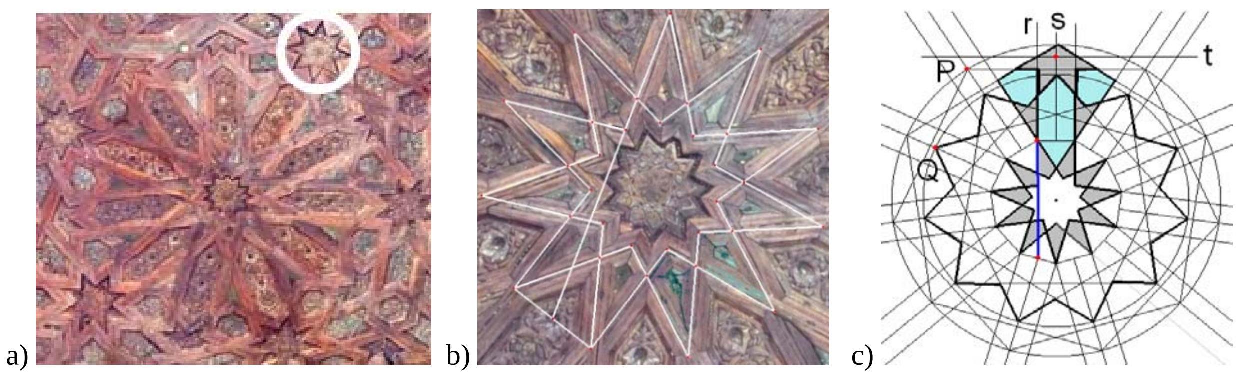
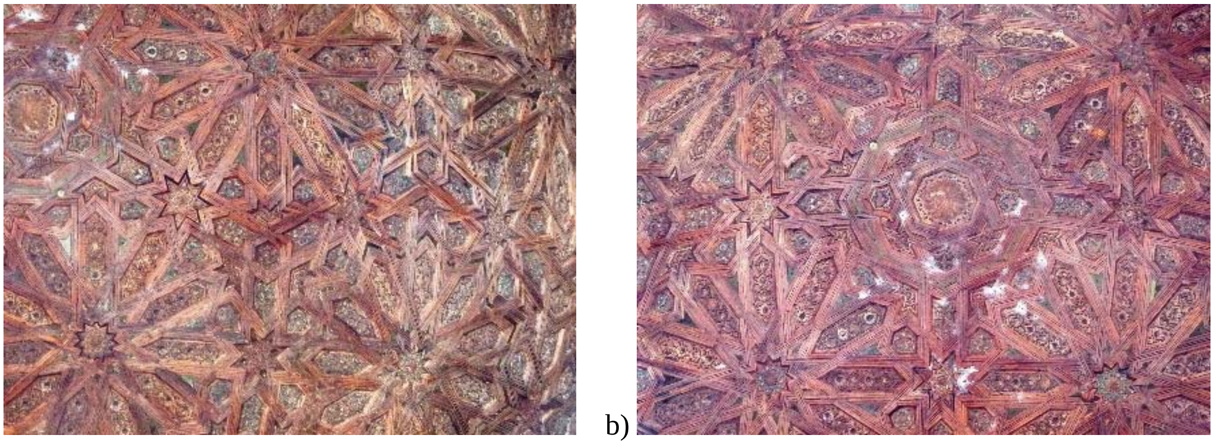


Figure 3: *a) The rosette; b) the central star of the rosette; c) scheme of the construction*

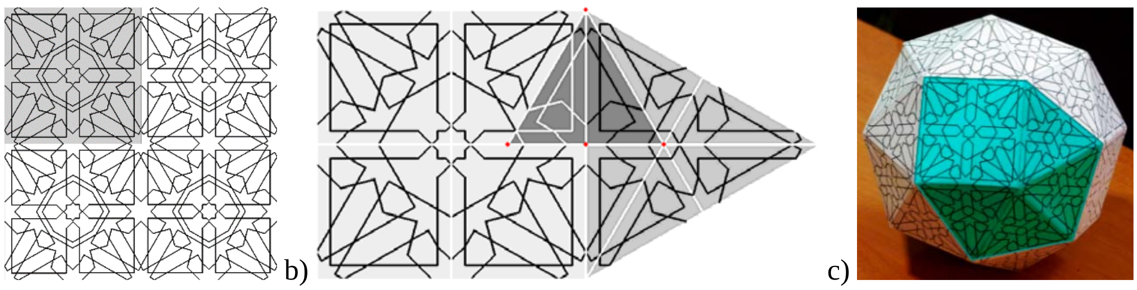
Note that the size of the rosette and the length of the ‘*petals*’ can be chosen freely, but the angles of its elements are constant and only determined by the angles of the central (11/4.5)2 star. That is, for any position of the points *P* and *Q* the acute angle of the (11/2)2 star is $7\pi/11$rad $\approx 114.5^{\circ}$, while it is $5\pi/11$rad $\approx 81.8^{\circ}$ for the (11/3)1 star. Therefore, in the ideal pattern, the small octagonal stars cannot be regular. Still, in the real pattern, the sides of the (11/3)1 star look as if they were perpendicular (Figures 3a and 4a), and so the octagonal star was probably designed as a (8/3)1 star. Figure 4 shows details of the triangle determined by three adjacent rosettes and the center of the pattern (from the western pavilion).

The preceding construction explained the pattern from a mathematical point of view. The last observations suggest it is not realistic to suppose this was the way in which the craftsmen actually made their construction. In addition, there is no evidence the hendecagonal near miss polyhedron was known after all. And so, here is a second construction procedure, which is perhaps more in line with Islamic woodwork techniques. The cylindrical vault ceiling of the Sala de la Barca in the nearby Comares Palace in Alhambra displays exactly the same oblong octagon and the same small (8/3)1 star together with large 12-stars (see [9] and Figure 5a) deriving from a flat Euclidean pattern. If we turn the 12-pointed stars into 11-pointed stars, then the plane turns into a polyhedral shape. Thus the triangle may be generated by reflection and rotation of a piece of the unit square of the Barca pattern (Figures 4a, 5a and 5b). This implies that the snub cube could be a *natural* polyhedral model for the practical pattern (Figure 5c). The center of the spatial rosettes is now set in each one of the vertices of this polyhedron, which is in fact the dual polyhedron of the pentagonal icositetrahedron. Each vertex figure is formed by grouping a part of the square (three angles of 30º) and four equilateral triangles (four times two angles of 30º).



a) b)

Figure 4: *details of the ceiling of western pavilion.*



a) b) c)

Figure 5: *a) Pattern of the ceiling of Sala de la Barca; b) pattern of the faces of the snub cube; c) polyhedral model based in the snub cube.*

The Nasrid woodworkers were skilled in covering flat, polyhedral and curved surfaces by elements derived from constructible regular polygons, and thus it seems realistic that the decorations of the domes were based on preliminary sketches such as those presented here. This is only speculation, but it is pleasant trying to discover how the artisans of the Alhambra made an early example (the earliest?) of a practical realization involving (parts of) the snub cube and a near miss eleven-sided polyhedron.

References

[1] *Arte Historia* (a Spanish art site): http://www.artehistoria.com/v2/obras/20778.htm

[2] B. L. Bodner, *The Eleven–Pointed Star Polygon Design of the Topkapı Scroll*, Bridges Conference Proc. 2009, 147-154.

[3] C. S. Kaplan, *Near Misses Project*, http://www.cgl.uwaterloo.ca/csk/projects/nearmisses/

[4] C. S. Kaplan, *Computer Generated Islamic Star Patterns*, Bridges Conference Proc. 2000, 105-112.

[5] C. S. Kaplan, *Islamic star ball with 11-pointed stars*, http://shpws.me/FNjU

[6] B. Pavón Maldonado, *La cúpula del Islam Occidental. Estructuras, decoración y símbolo*: http://www.basiliopavonmaldonado.es/Documentos/Cupulas.pdf.

[7] The Court of the Lions, http://www.alhambra-patronato.es/index.php/Patio-de-los-Leones-Fuente-Surtidor/164+M5d637b1e38d/0/

[8] Victoria and Albert Museum, London, http://www.vam.ac.uk/users/node/17511

[9] D. Wade, *Pattern in Islamic Art*, http://patterninislamicart.com/archive/main/7/spain/spa1621x