Sections of Coxeter Orbihedra

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Abstract

We study sections of high dimensional polytopes whose vertices form the orbit of a Coxeter group, and create "scans" of such polytopes in order to graphically visualise them for educational and public engagement purposes.

Introduction

Geometry in higher dimensions is a fascinating and challenging problem that has attracted the attention of mathematicians and scientists for a long time. Coxeter, in his celebrated book *Regular Polytopes* [1], presented a systematic study of regular polytopes, and introduced the concept of *Coxeter group* in order to study their symmetry properties. In the finite case, Coxeter groups correspond to groups generated by *reflections* through hyperplanes; familiar examples are the dihedral groups in the plane and the symmetry groups of the platonic solids.

Group theory is a fundamental tool in mathematics to describe objects with symmetry. A group of isometries G in the plane or in space acts as a collection of rigid motions; given a point x, the set $\mathcal{O} = \{g \cdot x : g \in G\}$ is called the *orbit* of x under G [2], and the convex hull of $\mathcal{O}$ defines a G*-orbihedron* [3]. In this paper we visualise such polytopes by "cutting" them through certain planes perpendicular to symmetry axes, thus obtaining *sections* of the orbihedra. Such sections are lower dimensional polyhedra which retain some symmetry properties of the original polyhedron.

A great challenge is to explain higher dimensional geometry to a wider audience, even beyond the mathematical community. To this aim, we define in this paper what is meant by a "scan" of a G-orbihedron $\mathcal{P}$: if we fix a direction perpendicular to some symmetry axes of $\mathcal{P}$, then we can vary continously the plane which cuts $\mathcal{P}$ along this direction, thus constructing a continous family of sections of $\mathcal{P}$. This process of "slicing" or "scanning" allows a nice visualisation of $\mathcal{P}$ with some particular symmetry properties. This can be in principle applied to any higher dimensional polytope; however in the case of G-orbihedra we can formulate the problem in terms of root systems and weights, which gives a better understanding of the geometry of the scans.

Coxeter groups, root systems and weights

Coxeter groups have been studied extensively and constitute an important field of research in abstract algebra [4]. In the finite dimensional case, they correspond to groups generated by *reflections*. Specifically, a

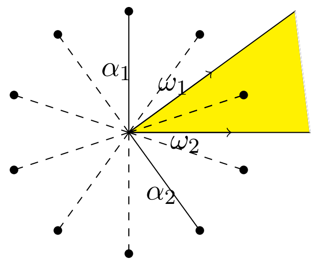

Figure 1: *The root system, simple roots and fundamental weights for the dihedral group $\mathcal{D}_{10}$. The area highlighted corresponds to a patch of the fundamental chamber for the orbits of $\mathcal{D}_{10}$.*

reflection $r_{\boldsymbol{\alpha}}$ associated with a vector $\boldsymbol{\alpha} \in V$, with V a finite dimensional vector space with inner product $\langle , \rangle$, is the linear operator given by $r_{\boldsymbol{\alpha}}(\boldsymbol{v}) = \boldsymbol{v} - \frac{2\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle}{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle}\boldsymbol{\alpha}$, with $\boldsymbol{v} \in V$.

A *root system* ϕ is a finite set of vectors such that $\phi \cap \mathbb{R}\boldsymbol{\alpha} = \{\boldsymbol{\alpha}, -\boldsymbol{\alpha}\}$ and $r_{\boldsymbol{\alpha}}\phi = \phi$ for every $\boldsymbol{\alpha} \in \phi$. A finite Coxeter group G corresponds then to a group generated by reflections $r_{\boldsymbol{\alpha}}$ associated with a root system ϕ. It can be proved that G can be generated by a finite number of reflections in ϕ, denoted by $\Delta = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k\}$, and called *simple roots*. The set $D^+ = \{\boldsymbol{x} \in \mathbb{R}^k : \langle \boldsymbol{x}, \boldsymbol{\alpha}_i\rangle \geq 0,\ \boldsymbol{\alpha} \in \Delta\}$ is called the *dominant chamber* of G and is such that every orbit of G contains exactly one point $\boldsymbol{x}^*$ in D^+, referred to as the *dominant point* of the orbit. Finally, it is convenient to introduce the basis $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k$ of *fundamental weights* defined by $\frac{2\langle \boldsymbol{\alpha}_i, \boldsymbol{\omega}_j\rangle}{\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i\rangle} = \delta_{ij}$, fo $i, j = 1, \ldots, k$, where δ_{ij} is the Kronecker delta. It is easy to check that a point $\boldsymbol{x}$ belongs to the dominant chamber if and only if its coordinates in the ω-basis are all non-negative. In Figure 1 we show the root system, simple roots and fundamental weights for the dihedral group $\mathcal{D}_{10}$.

Sections and scans of Coxeter orbihedra

Let us consider a Coxeter group G with simple roots $\Delta = \{\boldsymbol{\alpha}_i, \ldots, \boldsymbol{\alpha}_k\}$ and fundamental weights $\{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k\}$. Given a point $\boldsymbol{x}$ in the fundamental chamber D^+, we consider the orbit $\mathcal{O}_G(\boldsymbol{x})$; its convex hull defines a G-orbihedron $\mathcal{P}$. Let us choose a subset $\Pi = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ of Δ, consisting of m roots, and let $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m$ be the corresponding weights. Let $\boldsymbol{c} = (c_1, \ldots, c_m)$ be a vector in $\mathbb{R}^m$, and let us define the affine subspace $H_{\boldsymbol{c}}$ given by

$$H_{\boldsymbol{c}} := \{\boldsymbol{v} \in \mathbb{R}^k : \langle \boldsymbol{v}, \boldsymbol{\omega}_i\rangle = c_i, i = 1, \ldots, m\}.$$

The dimension of $H_{\boldsymbol{c}}$ is $d := k - m$. It is straightforward to see that $H_{\boldsymbol{c}}$ is invariant under the reflections r_{α_i}, with $\alpha_i \notin \Pi$; in fact, we have

$$\langle r_{\boldsymbol{\alpha}_j}(\boldsymbol{v}), \boldsymbol{\omega}_i\rangle = \langle \boldsymbol{v} - 2\frac{\langle \boldsymbol{\alpha}_j, \boldsymbol{v}\rangle}{\langle \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j\rangle}\boldsymbol{\alpha}_j, \boldsymbol{\omega}_i\rangle = \langle \boldsymbol{v}, \boldsymbol{\omega}_i\rangle - 2\frac{\langle \boldsymbol{\alpha}_j, \boldsymbol{v}\rangle}{\langle \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j\rangle}\underbrace{\langle \boldsymbol{\alpha}_j, \boldsymbol{\omega}_i\rangle}_{=0} = \langle \boldsymbol{v}, \boldsymbol{\omega}_i\rangle = c.$$

A *section* of $\mathcal{P}$ with respect to $\boldsymbol{c}$ is the set $\mathcal{Q}_{\boldsymbol{c}} := \mathcal{P} \cap H_{\boldsymbol{c}}$. $\mathcal{Q}_{\boldsymbol{c}}$ is a polytope of dimension d invariant under the Coxeter group $\tilde{G}$ generated by $r_{\boldsymbol{\alpha}_j}$, with $\boldsymbol{\alpha}_j \notin \Pi$, which is a subgroup of G. With this setup, we can define a *scan* of $\mathcal{P}$ with respect to Π the function $S_\Pi : \mathbb{R}^m \to \mathbb{P}(\mathbb{R}^d)$, the power set of $\mathbb{R}^d$, given by $\boldsymbol{c} \mapsto \mathcal{Q}_{\boldsymbol{c}}$. We focus here on the case $m = 1$, i.e. sections of codimension 1. In this case, we choose a weight $\boldsymbol{\omega}^*$ and remove it from the set of weights. Hence, the scans $S(c)$ are functions parameterised by $c \in \mathbb{R}$. Combinatorially, we can associate to a polytope $\mathcal{P}$ its *face lattice* $l(\mathcal{P})$, i.e. the set of inclusions of its faces (of various dimensions) [5]. We thus say that $c^* \in \mathbb{R}$ is a *critical point* for the scan of a polytope $\mathcal{P}$ if $l(S(c))$ changes for $c = c^*$. Intuitively, a change in the sections occurs when the hyperplane H_c "hits" a vertex of $\mathcal{P}$.

We present here two examples of scans, for the case of the Coxeter group H_4, which is associated with generalised icosahedral symmetry in four dimensions [4]. Specifically, we consider two polytopes with

H_4 symmetry: the 120-cell and the 600-cell, which correspond to the dodecahedron and the icosahedron in four dimensions, respectively [1]. We construct various sections of these polytopes (see Figure 2 and 3), by removing one suitable root to the the root system of H_4, in such a way that the resulting polyhedra retain icosahedral symmetry (cf.[6] for more details). Notice that, while they all share the same symmetry group, the sections have in general different combinatorial and geometrical properties, and various critical points occur during the scan. These snapshots can be used to create animations of the scans of polytopes, thus providing a concrete tool to "grasp" higher dimensional objects.

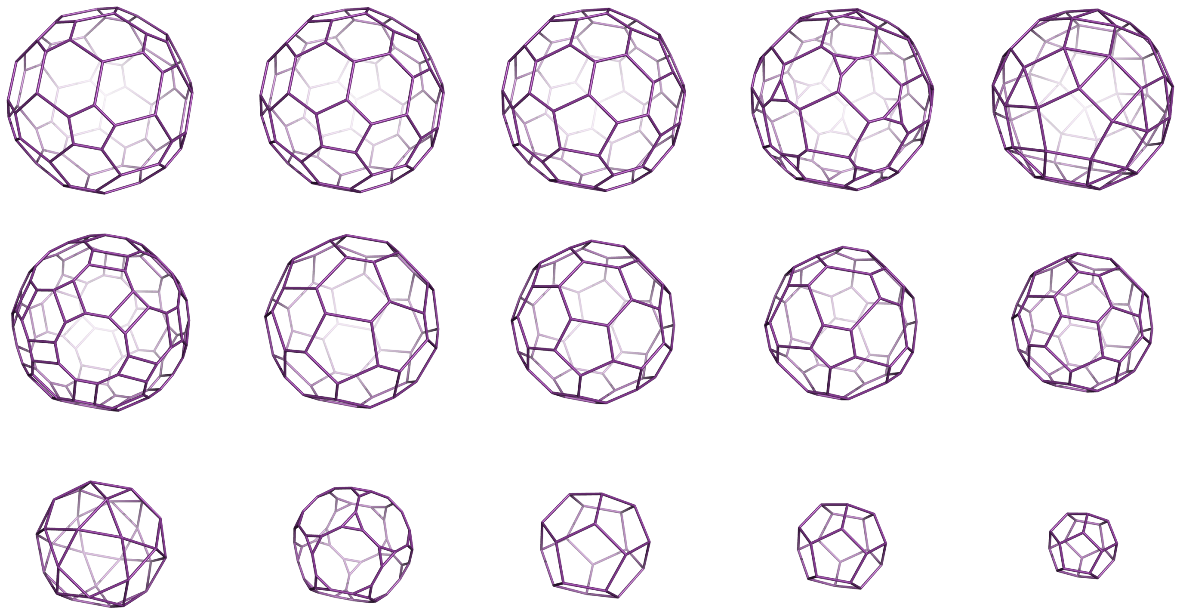

Figure 2 : *Snapshots of the scan of a four-dimensional 120-cells: all the sections retain icosahedral symmetry.*

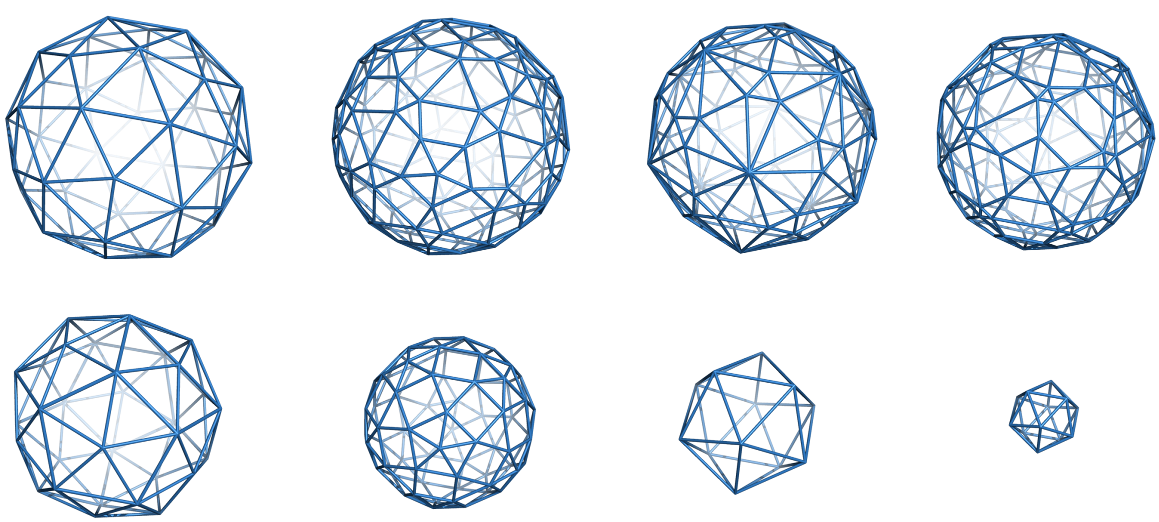

Figure 3 : *Snapshots of the scan of a four-dimensional 600-cells: as for the case of 120-cells, we consider sections with icosahedral symmetry.*

Finally, we briefly point out that it is possible to describe geometrically the sections of Coxeter orbihedra of codimension 1 as an intersection of polytopes with prescribed symmetry properties. We give an example in Figure 4: an icosahedron is cut along a five-fold symmetry axis, resulting in a polygon with symmetry group $\mathcal{D}_5$, which can be seen as the intersections of two pentagons. Algebraically, this can be explained by the fact that any convex polytope $\mathcal{P}$ can be described as an intersection of hyperplanes [5]; in the case of Coxeter groups, these can be written in terms of the fundamental weights $\{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k\}$. By removing one weight $\boldsymbol{\omega}^*$, we construct the corresponding hyperplane H_c which cuts the polytope, and project the bounding hyperplanes into H_c, resulting in a set of lower dimensional polytopes whose intersection is $H_c \cap \mathcal{P}$. Such polytopes are face-transitive with respect to the group $\tilde{G}$ as defined above, and they arise by considering $\tilde{G}$-orbits of bounding hyperplanes of $H_c \cap \mathcal{P}$. We are working on a formal proof of this, as well as on the combinatorial and geometrical properties of scans with codimension greater than one.

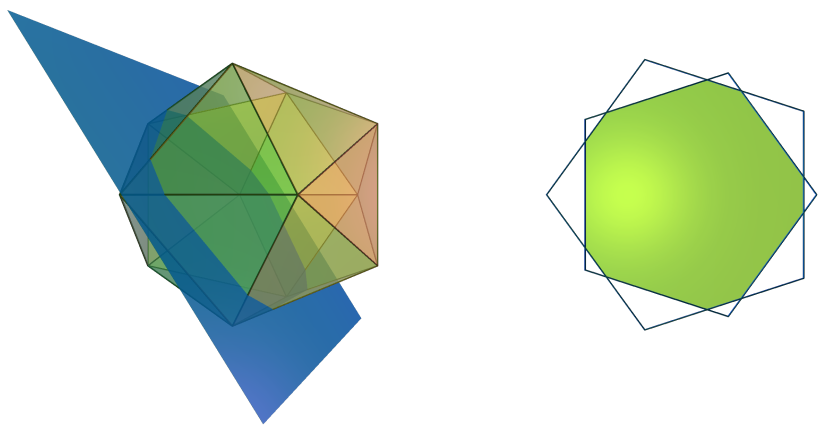

Figure 4: *Section of an icosahedron along a five-fold axis: the result (on the right) is a polygon with five-fold symmetry, which can be described geometrically as the intersection of two pentagons.*

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