Sculpturing Surfaces with Cartan Ribbons

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Abstract

Using the concepts of *Cartan development* and *rolling* from differential geometry we develop a method for sculpturing any surface with the use of Cartan ribbons.

Peeling and Pasting

When peeling an apple or a potato you may steer the peeler along any chosen curve. Examples are shown in Figure 1 together with the corresponding peelings. Here we initiate a geometric study of this operation and

Figure 1 : *Examples of steering curves and corresponding peelings of a potato and an apple*

its reverse: we construct the flat ribbons that can be pasted onto a given surface so that they approximate the surface to first, tangential, order along one or more curves on the surface. We can imagine that once the flat ribbons have been mathematically described, they can be cut out of paper, a thin metal plate, or any other flat, bendable material. The flat ribbons are constructed along the so-called *Cartan development* of the given surface curve – that is why we call these ribbons *Cartan ribbons*.

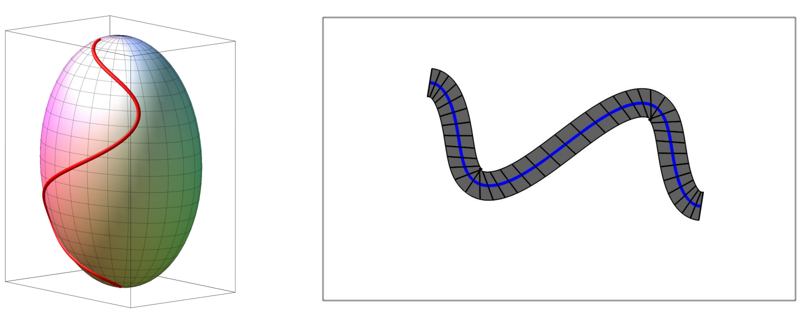

Figure 2 : *The result of a peeling along a given curve on an ellipsoid*

This pasting technique, the sculpting, with Cartan ribbons works along any given curve on the given sculpture surface as long as the curve just avoids the asymptotic directions of the surface. A good starting point for a precise discussion of the method and of this condition is the seminal paper by K. Nomizu [8]. Nomizu gives a kinematic interpretation of the Cartan development via the rolling of the given surface on a plane – see also the recent work by M. G. Molina, [7]. We can imagine that the designated curve on the surface is covered with wet paint. We then roll the surface on a plane without skidding and twisting, and in such a way that the points of contact between the surface and the plane are precisely along the given curve. Then the wet curve on the surface will trace out and paint a new curve on the plane. This new planar curve is precisely the Cartan development of the original surface curve. One intuitive way of understanding the procedure is then to divide it into three steps: (1) the given curve on the surface is rolled into the plane as described above; (2) the resulting Cartan development curve is extended to a strip, a Cartan ribbon, in the plane; and finally (3) the ribbon is rolled back along the development curve onto the surface. Using an ellipsoid to replace the potato, the three steps are illustrated in Figures 3, 4, and 5. The resulting flat peeling along the given curve is shown on the right in Figure 2, where the transversal rulings (like railroad sleepers) along the center curve are also shown. They represent the lines around which the Cartan ribbon must be bent in order to fit onto the surface along the given designated curve. The precise angle function θ of the rulings w.r.t. the curve tangent is discussed in the mathematics section below, see also [9] for a thorough discussion and generalizations.

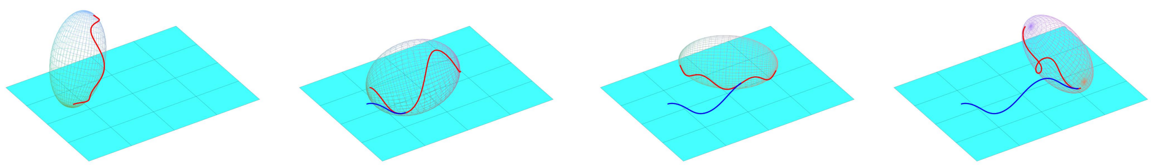

Figure 3 : *Step 1: Generating the Cartan development curve via rolling of the surface*

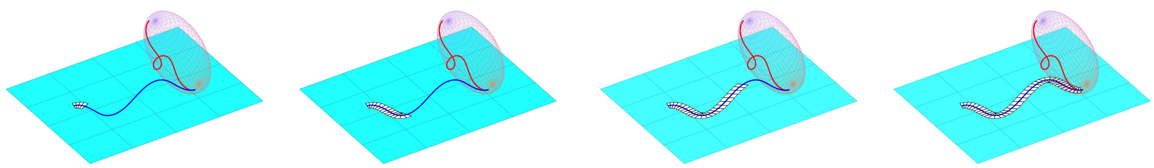

Figure 4 : *Step 2: Ribbonizing the Cartan development curve via the ruling angle function θ*

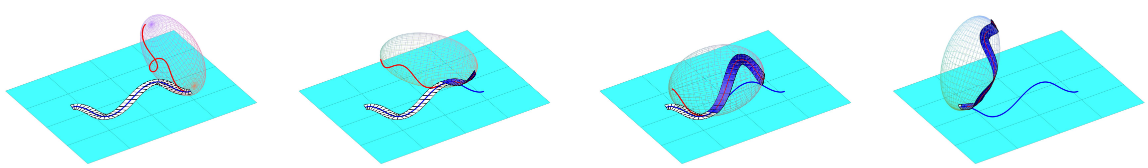

Figure 5 : *Step 3: Pasting the Cartan ribbon onto the surface via back-rolling*

This procedure clearly paves the way for a novel technique of approximating any surface or sculpture with one or more intrinsically flat Cartan ribbons. See [10, 6] for modern industrial applications of similar techniques. We can extend the ribbon until it intersects the extension of another ribbon (or itself); such surface approximations are illustrated in Figures 6 and 7.

From an artistic point of view the created surfaces have a surprising appearance. They contain wedges (ridges) as well as singly curved piecewise smooth and regular surfaces. They can be fabricated artistically in *coarse grained* versions creating unique surfaces with many obvious applications, e.g., lamp shades, chairs, roofs, facades, and the design of clothing and artworks, see [4, 5]. Or they can be *fine grained* in which case the underlying (typically *doubly curved*) surfaces become almost perfectly approximated by the *flat* Cartan ribbons as seen in the Figures 6 and 7. An interesting bonus is that the method also can be used to transform different surfaces into each other by using the same set of Cartan developments only with different angle functions θ for the rulings. Artistically, it is thus possible to make interesting intermediate versions of two approximated surfaces.

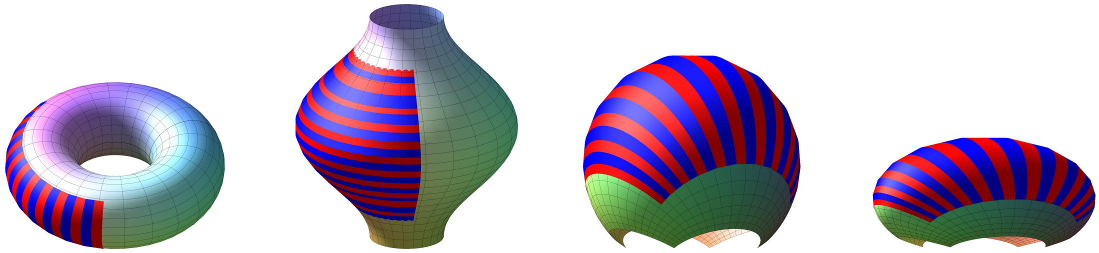

Figure 6 : *Surfaces with systems of approximating Cartan ribbons*

The Mathematics

The main geometric question in our setting is how precisely to cut the Cartan ribbons and how to find and describe the bending, i.e., the angle function θ, of a Cartan ribbon that will actually fold it onto the surface along the given surface curve. We answer these questions by surveying here the explicit recipes for both the cutting and for the bending. Details for this particular application as well as generalizations will appear in [9]. We also refer to [1, 2] for similar geometric results and other applications of ribbon geometry.

For the precise discussion of this recipe we need the notions of normal curvature κ_n, geodesic curvature κ_g, and geodesic torsion τ_g of the given designated curve on the surface. They are found most easily via the so-called Darboux frame adapted to the surface along γ. It consists of three orthogonal unit vector fields $\{e, h, N\}$ along γ. Here $N(t)$ denotes the unit normal vector to the surface at $\gamma(t)$ and $e(t) = \gamma'(t)/v(t)$ is the unit tangent vector of γ (where we have denoted the speed of γ by $v(t) = \|\gamma'(t)\|$), so that finally $h(t) = N(t) \times e(t)$ completes the orthonormal Darboux frame, see [3]. Then $v(t)\,\kappa_n(t) = e'(t) \cdot N(t)$, $v(t)\,\kappa_g(t) = e'(t) \cdot h(t)$, and $v(t)\,\tau_g(t) = h'(t) \cdot N(t)$. We can now state the result of Nomizu as follows:

Theorem [8, Nomizu, Theorem 2 p. 630] *Let γ denote a smooth regular curve on a given surface M. Suppose that the normal curvature $\kappa_n(t)$ is never zero along γ. Then there exists a unique rolling Rol_t of M on the (x, y)-plane (without skidding and without spinning) such that $\eta(t) = \mathrm{Rol}_t(\gamma(t))$ (with $\eta(0) = (0, 0)$ and $\eta'(0) = (0, \|\gamma'(0)\|)$) is the locus of points of contact during the rolling. The curve η is the Cartan development of the given curve γ.*

The condition that *the curve just avoids the asymptotic directions of the surface* alluded to above is here encoded into the assumption that *the normal curvature $\kappa_n(t)$ is never zero along γ*. Such a condition is needed: a non-flat ruled surface, for example a hyperboloid of one sheet, cannot roll in the direction of its rulings, so the curve γ on the surface should never have a tangent parallel to such a ruling. Correspondingly, the Cartan ribbon construction also needs special care when $\kappa_n(t)$ is 0:

Proposition [9] *The unique flat (but bent) Cartan ribbon along γ which has the same normal field N as the surface M is well defined for $\kappa_n(t) \neq 0$. It is the ruled, developable, surface determined by the ingredients $e(t)$, $h(t)$, $\kappa_n(t)$, and $\tau_g(t)$ as follows: $r(t, u) = \gamma(t) + u \cdot \left(\frac{\tau_g(t)}{\kappa_n(t)} \cdot e(t) + h(t)\right)$, where $u \in [-w, w]$.*

In this representation the ribbon has constant width $2w$; it can easily be modified to variable widths $w_+(t)$ to the left hand side (the $h(t)$ side) of γ and $w_-(t)$ to the right hand side (the $-h(t)$ side) of γ just by substituting the u interval by $u \in [-w_-(t), w_+(t)]$. As is evident from the examples in Figures 6 and 7 we need to apply this type of restriction so that the approximating ribbons line up along their edges.

The tangent vectors $\gamma'(t)$ and $\eta'(t)$ have the same coordinates with respect to corresponding parallel frames along the curves on M and in the plane, respectively. The angle function θ of the rulings is therefore determined by the angle between the vector $(\tau_g(t)/\kappa_n(t)) \cdot e(t) + h(t)$ and the tangent vector $e(t)$, i.e., $\cot(\theta(t)) = \tau_g(t)/\kappa_n(t)$. The bending angle variation is illustrated on the right in Figure 2. The angle is well-defined and 0 precisely when $\kappa_n(t) = 0$ which must be avoided, because in such a case the ruling is directed in the tangent direction of the curve and the given parametrization of the ribbon is thence not regular. In this parametrization, then, the condition for regularity is the same as Nomizu's condition for the existence of a rolling, namely $\kappa_n(t) \neq 0$. Moreover, in order to obtain regularity of the ribbon with a finite width, the width to the *curvature side* of the center curve must also be less than $1/\kappa_g(t)$.

It follows in particular from Nomizu's theorem that the tangent vectors $\gamma'(t)$ and $\eta'(t)$ have the same length for all t and that the two curves have the same geodesic curvature function κ_g on M and in the plane respectively. This latter property means that if we are given one of the curves, then the other curve can be explicitly constructed by solving the ordinary differential equation system that produces the curve from its geodesic curvature function, see [3]. In this way the curve and its development are dual constructions – the given curve may be considered and re-constructed on the surface in this way as the *anti-development* of the Cartan development curve. We note in particular, that in the case of M being a sphere of any radius ρ, then $\kappa_n(t) = 1/\rho$ and $\tau_g(t) = 0$ along every curve in M, so we get $\theta(t) = \pi/2$ for all t. This makes it particularly simple and easy to construct the ribbons along curves on spheres, see Figure 7.

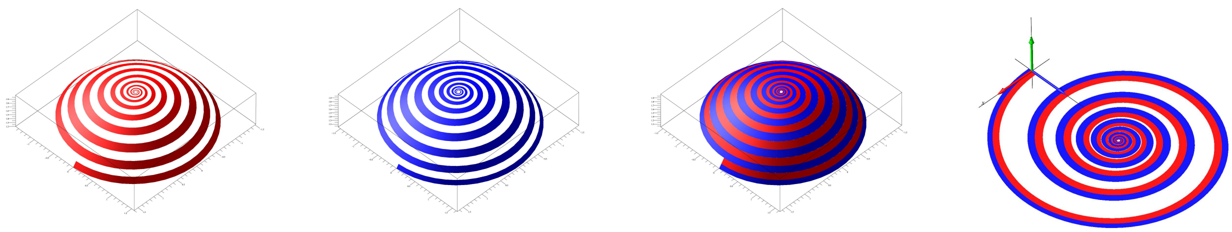

Figure 7 : *A Cartan ribbon (with controlled widths $w_{\pm}$) covers the spherical cap along a given spiral*

Acknowledgement. The authors would like to thank the referees for their careful reading of the manuscript.

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