A Musical Polyhedron Updated for the 21st Century

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Abstract

In the 19th century August Möbius described a musical torus constructed on the chromatic scale illustrated with two-dimensional diagrams. In the 20th century A. Harry Wheeler realized Möbius’s concept in a three-dimensional model. With the availability of 3D printers at the beginning of the 21st century, the construction of this musical polyhedron can be reimagined. The flat-pattern models generated by the 3D model make the underlying geometry more accessible, and the new 3D model brings out sculptural qualities that were not as evident before.

Introduction

This is a story of mathematical model building combining music, math, and art that spans three centuries. It begins in the 19th century with August Möbius describing a figure based on harmonic relationships within the chromatic scale that could be used to create a polyhedral torus. In the 20th century Harry Wheeler would create this model in both paper and plastic relying on compass-and-straight-edge techniques. Using Wheeler’s model as a springboard, we can use 3D software and printers to reconceptualize the construction process and gain insight into Möbius’s original vision.

Möbius’s Concept

Möbius (1790-1868), the German mathematician best known for his discovery of the Möbius strip, described a three-dimensional polyhedral torus having 12 corners corresponding to the 12 tones of the chromatic scale, 24 triangular faces corresponding to the 12 major and 12 minor triads, and 36 edges corresponding to intervals of major thirds, minor thirds, and perfect fifths [1]. Möbius created a pattern of notes based on pitch names shown in Figure 1 to make the relationships clear. I have replaced the pitch names B and H common in Germany, Finland, and Scandinavia with the Anglo-American pitch names A# and B, respectively, to use the musical relationships more familiar to my students and colleagues.

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   C     E     G#    C
D#    G     B     D#
   A#    D     F#    A#
C#    F     A     C#
   G#    C     E     G#
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Figure 1: *Arrangement of pitch names.*

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   C     E     G#    C
D#    G     B     D#
   A#    D     F#    A#
C#    F     A     C#
   G#    C     E     G#
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Figure 2: *A major triad and a minor triad.*

Notes in each horizontal row are separated by major thirds as we move from left to right. Notes that fall in diagonal lines from upper left to lower right are separated by perfect fifths. Notes that fall in diagonal lines from upper right to lower left are separated by minor thirds. The structure is made even clearer if we look at the connections between adjacent rows. Downward-pointing triangles constructed from notes in adjacent rows form major triads that link one row to another. The first and second rows are linked by the major triads C E G, E G# B, and G# C D#. For example, a downward-pointing triangle, the major triad C E G, is outlined in Figure 2. Similarly, upward-pointing triangles constructed from notes in adjacent rows form minor triads linking one row to another. The first and second rows are linked by the minor triads C D# G, E G B, and G# B D#. An upward-pointing triangle, the minor triad G# B D#, is outlined in Figure 2.

Möbius envisioned each horizontal row as a triangular base. Each base is connected to another base by 3 major and 3 minor triads as shown in Figure 3. The interior equilateral triangle has vertices labeled with notes from the first row, C E G#. The outside equilateral triangle has vertices labeled with notes from the second row, D# G B. The inside and outside triangles are connected by the 3 major triads (the acute triangles) and 3 minor triads (the oblique triangles). If we imagine pulling up the center equilateral triangle out of the plane of the paper and stretching the 6 connecting triangles while the outside equilateral triangle stays fixed in the plane, we can envision an octahedron with two equilateral bases connected by 6 triangular faces. We will call this octahedron 1. Following Möbius's lead, we can construct octahedron 2 (Figure 4) from the second and third rows. Note that the outside triangle of octahedron 1 (D# B G) becomes the inside triangle of octahedron 2 with a 60° clockwise rotation from its original orientation; this rotation will appear in the 21st-century 3D construction. Octahedron 1 can be connected to octahedron 2 at their common base. Octahedron 3 can be constructed from the third and fourth rows, and octahedron 4 can be constructed from the fourth and fifth rows, the fifth row having the same notes as the first row. Octahedron 2 will connect with 3, 3 will connect with 4, and 4 will come full circle to connect with 1. More information about the mathematics of the polyhedral torus is available at the Smithsonian National Museum of American History website [2].

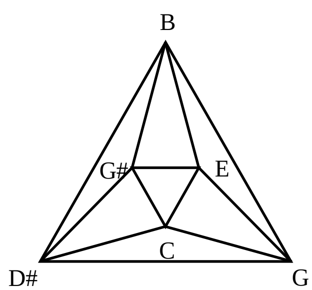


Figure 3: *2D perspective of octahedron 1.*

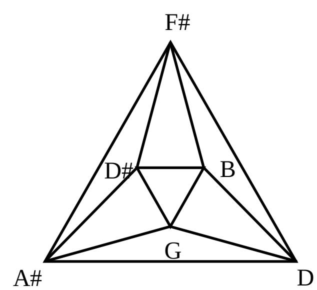


Figure 4: *2D perspective of octahedron 2.*

Wheeler's Model

Harry Wheeler (1873-1950) was an American mathematician who taught in high schools in Worcester, Massachusetts [3]. The mathematics collection of the Smithsonian National Museum of American History holds several models constructed by Wheeler, including plastic and paper models of the musical polyhedron shown on the Smithsonian Museum's website [2]. The Museum also holds Wheeler's original compass-and-straight-edge drawings dated July, 1939, for constructing the models. Figure 5 shows the flat paper pattern that Wheeler created [4]. The four sides of the polyhedral torus, labeled 1 through 4, are visible; each side has six triangles. What is missing

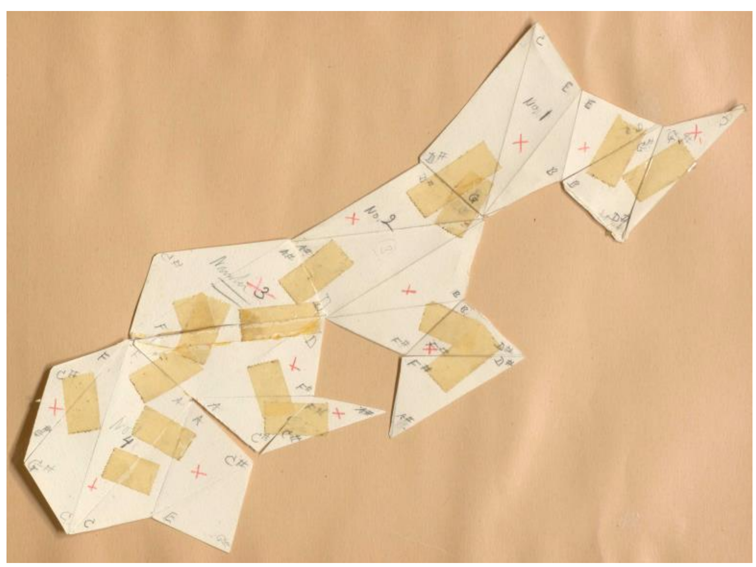


Figure 5: *Wheeler's model ready to be folded.*

are the equilateral triangles that he originally drew for the bases. A craftsman, Wheeler made his model with as few joins as possible. The bases of the octahedra are not visible when the sides of the polyhedron are connected because they are sandwiched between adjoining sides, so Wheeler eliminated them. Each of the four sides becomes a six-sided tube. Connecting the four tubes gives the 24-sided polyhedron.

The 21st Century

In constructing a 3D image of this musical torus, Wheeler's model was my starting point. After careful scrutiny of the photographs on the Smithsonian's website and scans of his paper and pencil constructions, I could see that the bases Wheeler omitted from his paper model would provide the framework for constructing the 3D model. I used the software program Rhinoceros 3D (version 5) to construct the virtual 3D model. Using a square in the xy-plane for the framework, I put an equilateral triangle orthogonal to the xy-plane in each corner as shown in Figure 6 (left). These are the four bases. The triangle in the northwest corner is Möbius's inner triangle in Figure 3, and the triangle in the northeast corner is the inner triangle in Figure 4. Note the 60° rotation around the framework of the northeast triangle relative to the northwest triangle, just as we observed going from Figure 3 to Figure 4. Now it is simple to join the appropriate vertices together to create the six triangular faces for each side of octahedron 1. Figure 6 (center) shows the connections between the first and second base to get a figure with six solid sides and two open sides. This makes octahedron 1 if we include the open sides, but it is really a six-sided tube just like Wheeler's. I constructed all four octahedra in order to create the flat patterns for paper construction, but I connected the four six-sided tubes to create the graphic version of the three-dimensional model shown in Figure 6 (right).

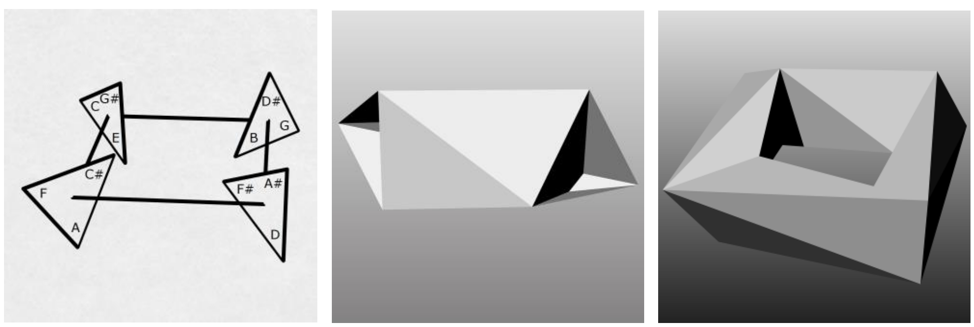


Figure 6: *Steps in construction of the graphic version of the three-dimensional model.*

Each octahedron constructed with Rhino3D was converted to an object file that became input for Pepakura Designer, version 3.1.8. Pepakura produces unfolded patterns from 3D models. After some manipulation I was able to produce a flat pattern for each octahedron, shown in Figure 7. The US letter-size patterns with instructions for assembly are available at http://blogs.jccc.edu/coneil. Figure 8 shows the photograph of the finished paper model. The key to assembly is matching and fastening the equilateral triangular bases to each other, perhaps not the most elegant method of making a model, but the process relates directly to Möbius's concept of linking base to base with major and minor triads. To get a good understanding of the geometry involved, nothing is better than constructing a model with one's own hands.

But the sculptural aspect of the torus is best experienced when holding and turning the solid model shown in Figure 9. Shapeways, a New York-based 3D printing service, printed the 3D model from the stereolithographic file created by Rhino 3D. The model, made from nylon, is approximately 12x12x5 cm.

Modern breakthroughs in technology make it possible for us to view classic ideas through a new lens. Just as Wheeler reconceptualized Möbius, here I have reconceptualized Wheeler's 20th-century vision via 21st-century technology. In this case 3D printing renews our appreciation of a 19th-century concept.

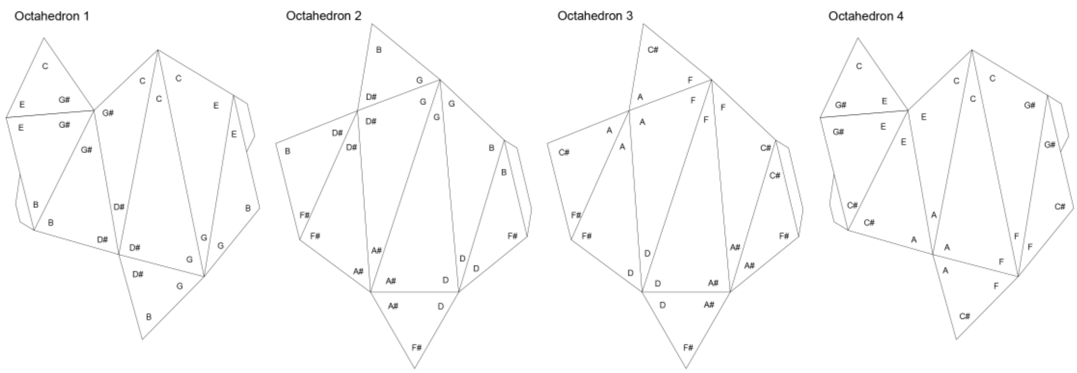
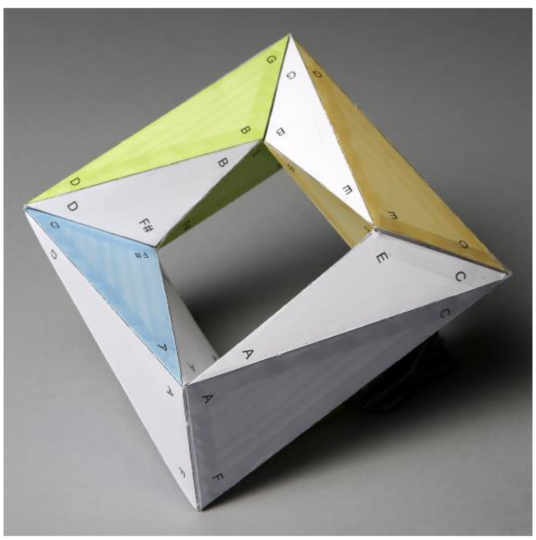


Figure 7: *Flat patterns for the musical polyhedron, ready to be cut and assembled.*



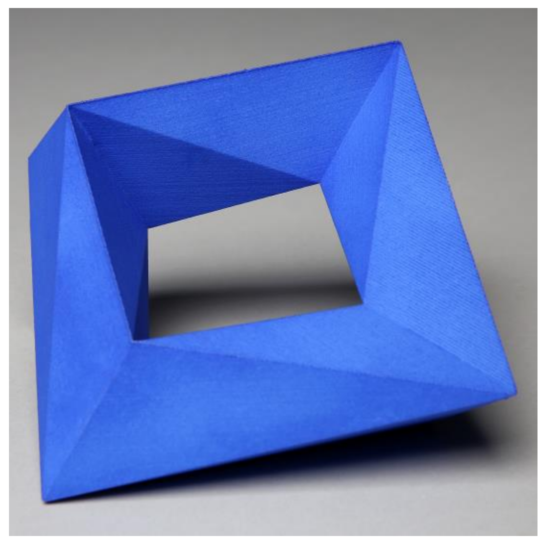


Figure 8: *Assembled paper musical polyhedron.* **Figure 9**: *3D printed musical polyhedron.*
Photographs by Susan McSpadden

Acknowledgments

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