Generalized Brunes Stars and System of Pythagorean Triples

Dmitri Kozlov
Research Institute of Theory and History of Architecture and Town-planning
Russian Academy of Architecture and Building Sciences
21-a 7th Parkovaya St. Moscow, 105264, Russia
E-Mail: kozlov.dmitri@gmail.com

Abstract

The paper describes a geometric way of constructing of right triangles with sides correspond to Pythagorean number triples as a further development of Tons Brunes eight-pointed star principle. An assumption is made that the geometric statement for Pythagorean triples might precede the algebraic one.

Eight-Pointed Star and Regular Square Grids

The eight-pointed star was introduced nearly 50 years ago by Tons Brunes as a reconstruction of ancient general method of division of square sides into 2, 3, 4 and 5 equal parts (fig. 1, *a*). Process of doubling and tripling of the basic 3×3, 4×4, and 5×5 square grids gives 6×6, 8×8, 9×9, and 10×10 grids [1]. The remaining 7×7 grid T. Brunes constructed by means of an approximation that he named "*sacred cut*". Nevertheless, methods for constructing of precise 7×7 grid can be found (fig. 1, *b*) [2].

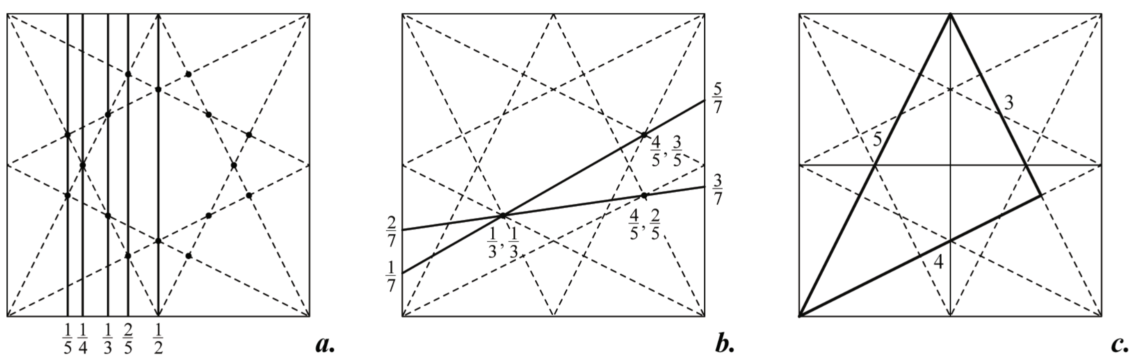

Figure 1: *Brunes star and division of square into 2, 3, 4, and 5 (a), 7 equal parts (b), and 3,4,5 right triangle (c).*

Because the lines of the Brunes star generate 3,4,5 right triangles and the star itself is based on 2×2 square grid (fig. 1, *c*), it is possible to suppose that other regular square grids may yield some other right triangles which sides relate as Pythagorean number triples.

Elementary Cases of 2×2 and 3×3 Square Grids

In order to find the relationship of sides of the right triangle *abc* generated by intersection of three diagonals of 1:2 rectangles in the square grid 2×2 one should notice that the hypotenuse *c* is the sum of the leg *a* and the distance *d*, that is equal to *b*/2, because *b* and *d* are the legs of the right triangle similar to the triangle with legs 2 and 1 (fig. 2, *a*). Consequently, the hypotenuse *c* is $a + b/2$ and, according to the Pythagorean theorem, the square of *c* is the sum of squares of *a* and *b*, that gives the next equation:

$$c^2 = a^2 + b^2 = \left(a + \frac{b}{2}\right)^2 = a^2 + ab + \frac{b^2}{4}.$$

From this equation it is easy to define that the relationship of the legs *a* and *b* equals to 3/4:

$$b = a + \frac{b}{4} \quad \rightarrow \quad a = \frac{3}{4}b \quad \rightarrow \quad \frac{a}{b} = \frac{3}{4}.$$

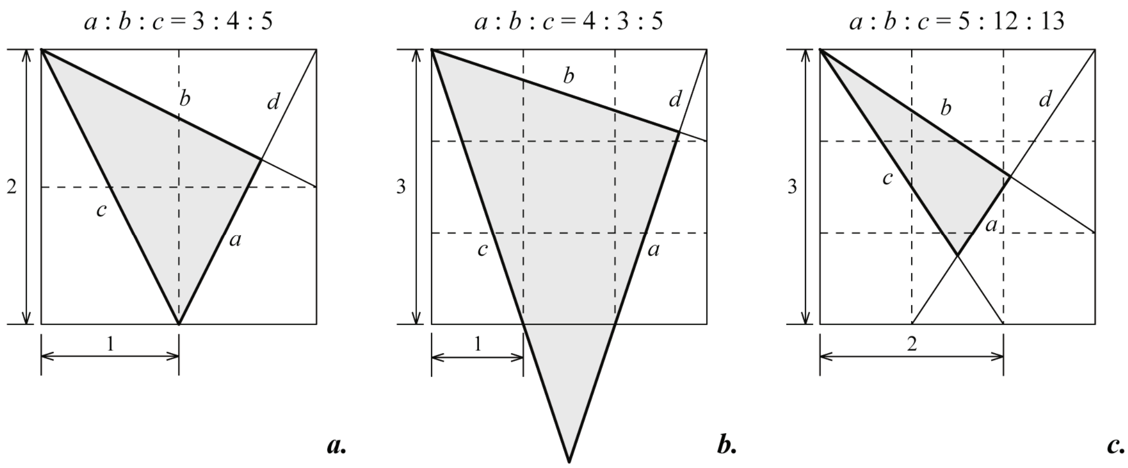

Figure 2: *Diagonals of 1:2 rectangles in 2×2 square give 3,4,5 triangle (a), diagonals of 1:3 rectangles in 3×3 square also give triangle 3,4,5 (b), and diagonals of 2:3 rectangles in 3×3 square give 5,12,13 triangle (c).*

By analogy with the 2×2 grid one can consider a 3×3 square grid. In this case it is possible to distinguish two type of rectangles, namely with 1:3 and 2:3 relationship of sides. The hypotenuse *c* of the right triangle *abc* that is generated by three diagonals of 1:3 rectangles (fig. 2, *b*) is the sum of the leg *a* and the distance *d* that is equal to $b/3$. At the same time the square of hypotenuse *c* is the sum of squares of *a* and *b*, that gives the equation from which it can be defined that the legs *a* and *b* relate as 4/3:

$$c^2 = a^2 + b^2 = \left(a + \frac{b}{3}\right)^2 = a^2 + \frac{2}{3}ab + \frac{b^2}{9},$$

$$b = \frac{2a}{3} + \frac{b}{9} \quad \rightarrow \quad \frac{2a}{3} = \frac{8}{9}b \quad \rightarrow \quad a = \frac{4}{3}b \quad \rightarrow \quad \frac{a}{b} = \frac{4}{3}.$$

In its turn the relationship of sides of the right triangle *abc* that is given by three diagonals of the 2:3 rectangles (fig. 2, *c*) gives another proportion. The hypotenuse *c* here is $a + 2b/3$ and the resulting equation demonstrates that the legs *a* and *b* relate as 5/12:

$$c^2 = a^2 + b^2 = \left(a + \frac{2b}{3}\right)^2 = a^2 + \frac{4}{3}ab + \frac{4b^2}{9},$$

$$b = \frac{4a}{3} + \frac{4b}{9} \quad \rightarrow \quad \frac{4a}{3} = \frac{5}{9}b \quad \rightarrow \quad a = \frac{5}{12}b \quad \rightarrow \quad \frac{a}{b} = \frac{5}{12}.$$

The right triangle with leg relationship of 5/12 is geometric interpretation of the second Pythagorean number triple, namely 5,12,13. The eight-pointed star that emerges as a result of overlapping of eight 5,12,13 triangles is not the “classical” Brunes star which vertices are lying on a square perimeter. The eight-pointed stars generated by intersection of eight rectangles’ diagonals in regular square grid which vertices are not lying on a square perimeter may be defined as “*generalized Brunes stars*”.

General Formula of Pythagorean Triples

It is possible to demonstrate that an arbitrary generalized Brunes eight-pointed star is composed of right triangles which sides are proportional to a Pythagorean number triple. One can consider a $v \times v$ square with three intersecting diagonals of $u \times v$ rectangles inscribed in it that forming a right triangle *abc* (fig. 3, *a*).

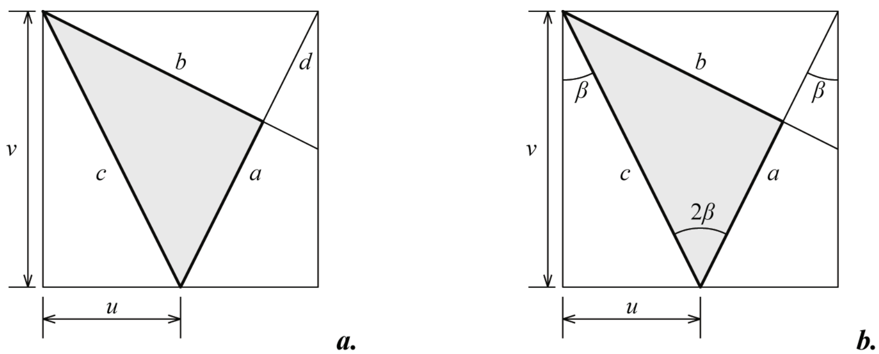

Figure 3: *Derivation of general formula for Pythagorean triples based on intersection of three diagonals of u×v rectangles in v×v square grid (a) and trigonometric approach to the same task (b).*

Like in the three previous particular cases, the hypotenuse c is equal to the sum $a + d$ and d relates to b as u relates to v. Therefore the equation $c = a + ub/v$ make it possible to find how a relates to b:

$$c^2 = a^2 + b^2 = \left(a + \frac{u}{v}b\right)^2 = a^2 + \frac{2u}{v}ab + \frac{u^2 b^2}{v^2},$$

$$b = \frac{2u}{v}a + \frac{u^2}{v^2}b \quad \rightarrow \quad \frac{v^2 - u^2}{v^2}b = \frac{2u}{v}a \quad \rightarrow \quad \frac{v^2 - u^2}{v}b = 2ua \quad \rightarrow \quad \frac{a}{b} = \frac{v^2 - u^2}{2uv}.$$

The result signifies that the legs a and b are proportional to the expressions $v^2 - u^2$ and $2uv$ correspondently. In this case the hypotenuse c is proportional to $u^2 + v^2$. It is much easier to find the relationship of a and b by means of trigonometry, taking into account that the angle between c and a is the double of angle between c and v (fig. 3, *b*):

$$\frac{a}{b} = \cot 2\beta = \frac{\cot^2 \beta - 1}{2\cot \beta} = \frac{\frac{v^2}{u^2} - 1}{2\frac{v}{u}} = \frac{v^2 - u^2}{2uv}.$$

The resulting formulas correspond to the general solution of the equation $a^2 + b^2 = c^2$ in integers. If u and v ($v > u$) are positive integer, they are not both odd and have no common factor, then the expressions $a = v^2 - u^2$, $b = 2uv$, and $c = u^2 + v^2$ satisfy primitive Pythagorean number triples [3].The generalized Brunes stars give a system of right triangles with sides proportional to both primitive and non-primitive

Pythagorean triples (fig. 4). Geometric interpretation of the algebraic conception of Pythagorean triples make it possible to distinguish non-primitive triangles as a relationship of sides of the $u \times v$ rectangles.

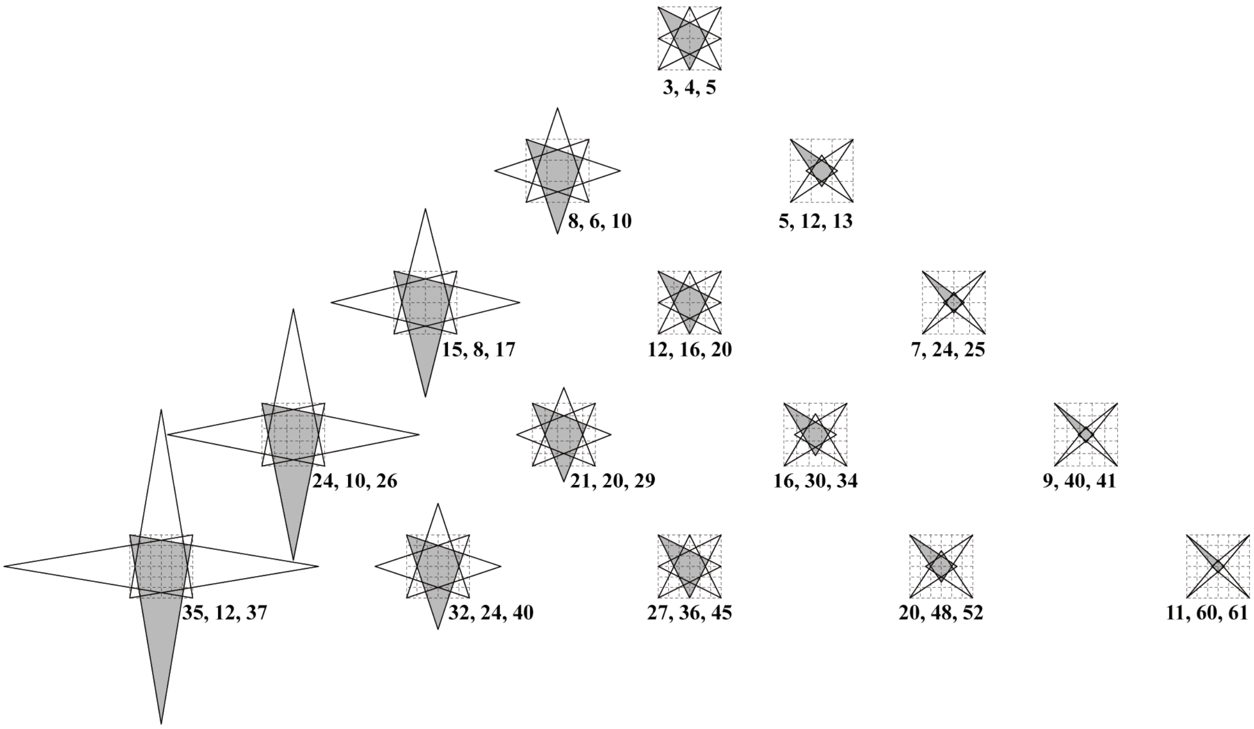

Figure 4: *Beginning of system of Pythagorean triples based on the generalized Brunes stars.*

The elementary square grids from 2×2 to 6×6 that can be easily constructed by means of the "classical" Brunes star (fig. 4) give eight first primitive Pythagorean triples that were known and practically utilized by most of ancient civilizations.

Conclusion

The Pythagorean triples were known at least since the times of ancient Babylon [4]. The origin of this knowledge traditionally attributed to the field of ancient abstract science such as algebra and number theory. Nevertheless, one may assume that the roots of Pythagorean number triples lay in geometry that was in all times close to visual arts, architecture, and crafts that required of different right triangles with integer sides to use them in practice.

The Brunes hypothesis about the ancient geometry and its figures both practical and symbolical may be naturally expanded to include the hypothesis of geometric origin of the Pythagorean triples.

References

[1] T. Brunes, *The* Secrets *of Ancient Geometry and its Use*, Rhodos, Copenhagen, 1967.
[2] D. Kozlov, *Eight-Pointed Star and* Precise *Construction of 7×7 Square Grid*, Proceedings of Bridges: Mathematics, Music, Art, Architecture, Culture, pp. 331 – 334, 2015.
[3] R. Courant, H. Robbins, *What is Mathematica?* Second. Ed., rev. by I. Stewart, Oxford University Press, 1996.
[4] O. Neugebauer, *The Exact Sciences in Antiquity*, Second Ed., Dover Publication, New York, 1957.