The Elevation of Coxeter's Infinite Regular Polyhedron 444444

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Abstract

In their book "La Divina Proportione" [1],[2], Luca Pacioli and Leonardo da Vinci described and illustrated an operation which you can apply to a polyhedron, called Elevation. Starting from Pacioli's basic idea, resulting in a second layer around a polyhedral shape, we can develop this idea further towards entwined double layer structures. Some of them are single objects, others appear to be compounds.

1. Introduction

1.1. La Divina Proportione. In their book "La Divina Proportione", published in 1509, Luca Pacioli and Leonardo da Vinci introduced the concept of Elevation, an operation that could be applied on the Platonic polyhedra as well as on the Archimedean polyhedra. The elevated versions of the first two Platonic solids, the tetrahedron and the octahedron, as they are drawn by Leonardo da Vinci, are shown in Figure 1 and 2.

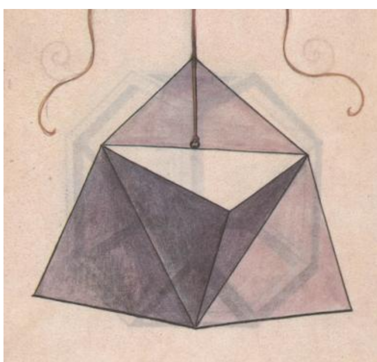


Figure 1: *Elevated Tetrahedron.*

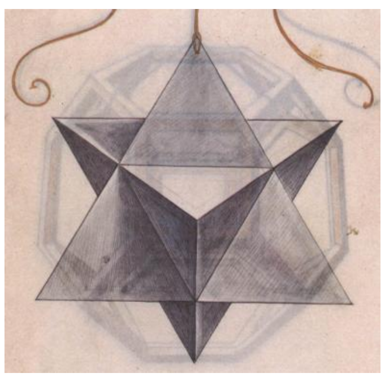


Figure 2: *Elevated Octahedron.*

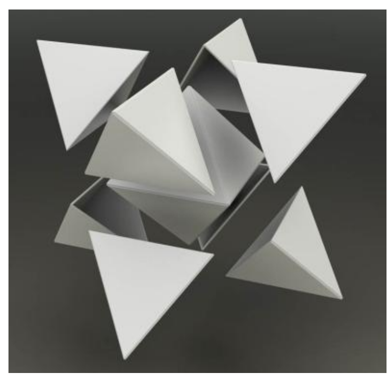


Figure 3: *Pacioli's description.*

1.2. Elevation. To understand what Elevation means we have to go to "La Divina Proportione" [3], chapter L, paragraph XIX.XX, where Pacioli describes the elevated version of the octahedron as follows: "And this object is built with 8 three-sided pyramids, that can be seen with your eyes, and an octahedron inside, which you can only see by imagination.". This means that the object is composed of 32 equilateral triangular faces of which 8 are hidden (Figure 3). Pacioli describes the process of elevation as putting pyramids, built with equilateral triangles, but without the bottom faces, on each of the faces of the polyhedra. The result of this operation is a double layered object and has many similarities with the stellated version of the octahedron. However "stellation" was introduced about a century later by Johannes Kepler in his "Hamonices Mundi" (1619). Kepler defined stellation for polyhedra, as the process of extending faces until they meet to form a new polyhedron. That means that the stellation of the octahedron consists of eight faces, just as the octahedron itself, instead of the 32 faces mentioned by Pacioli. Also the number of vertices is different: the stellation of the octahedron has 6 vertices (again the same number as the octahedron itself) whereas the elevated octahedron has 14 vertices (6 of the octahedron + 8 of the pyramids).

1.3. Vertex Figure. Let us have a closer look at the elevated octahedron (Figure 4). And especially at the 6 vertices of the original octahedron. After adding the 8 three-sided pyramids at each of the faces, we now have 12 triangles coming together in each of these vertices. Normally, in a polyhedron, just two faces join along any common edge. In the elevation however we have some edges where four faces come together. By the way Pacioli describes the process of elevation, it is very clear which faces are really joined. So when we look at the vertex figure around one of those vertices (the figure that we get when we "walk" around the vertex), we see that we have a first loop on the outer shell of the elevation (Figure 5), and a second loop on the inner shell (on the octahedron that is still inside), Figure 6. So the connection of the faces in these vertices can be described as follows: 33333333 (the faces of 4 pyramids surrounding a vertex) plus 3333 (the faces of the octahedron).

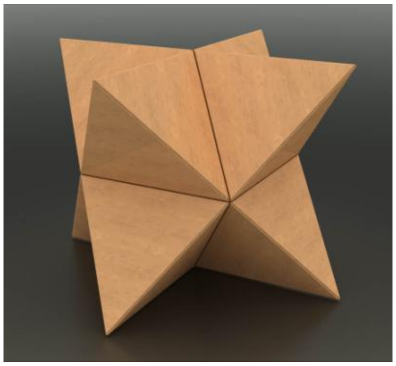


Figure 4: *Elevated Octahedron.*

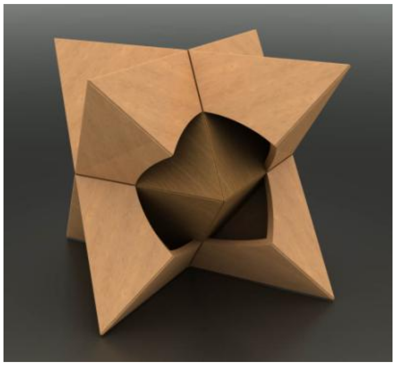


Figure 5: *Elevated Octahedron.*

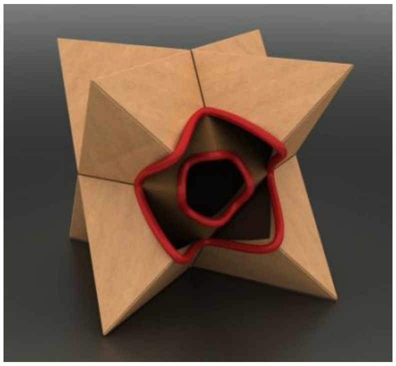


Figure 6: *Composition*

1.4. Elevation Element. Because of the special situation that four faces come together at each of the edges that meet in these vertices, we can take an alternative choice in pairing the faces. For instance, we can start our walk around the vertex on one of the faces of the octahedron and then step to an adjacent face of a three-sided pyramid.

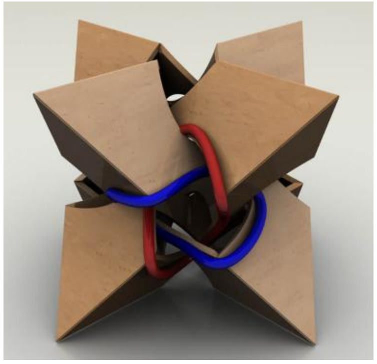


Figure 7: *Alternative loops.*

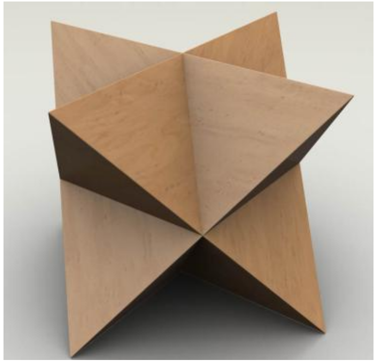


Figure 8: *Stella Octangula.*

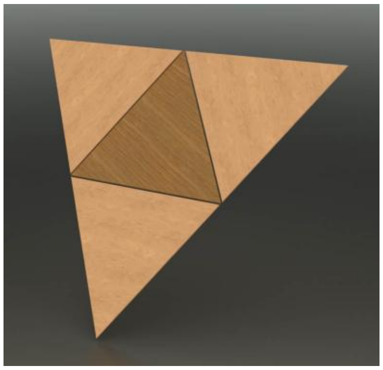


Figure 9: *Elevation element.*

When we continue our walk, we can make a loop-like the blue line in Figure 7. Again we end up with two loops around the vertex, but now the two loops are interlinked and the elevation splits up in two tetrahedron-like figures instead of one octahedron plus eight three-sided pyramids. It comes close to Kepler's stellated octahedron (Figure 8), but there is still a difference: we don't have real tetrahedra because the big triangles are in fact four joined triangles. Let us define this set of joined faces, one face of the basic polyhedron (in this case the octahedron) surrounded by faces of the elevation pyramids, one at each of the edges, as the "elevation element" (Figure 9).

2. Knots

2.1. Elevation of the Cube. After analyzing the double layer structure of Leonardo's elevated octahedron, the question might be whether we will see something similar at the other elevated polyhedra. In Figure 10 Leonardo's drawing of the elevated cube is shown. Luca Pacioli explains: "… it is enclosed by 24 triangular faces. This polyhedron is built out of 6 four-sided pyramids, together building the outside of the object as you can see it with your eyes. And there is also a cube inside, on which the pyramids are placed. But this cube can only be seen by imagination, because it is covered by the pyramids. The 6 square faces are the bottom faces of the 6 pyramids.". When we follow this explanation we can make the exploded view presented in Figure 11.

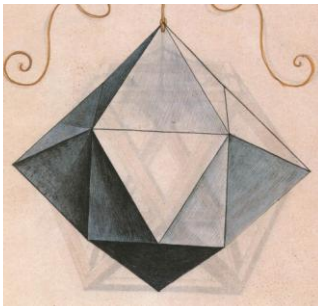


Figure 10: *Elevated Cube.*

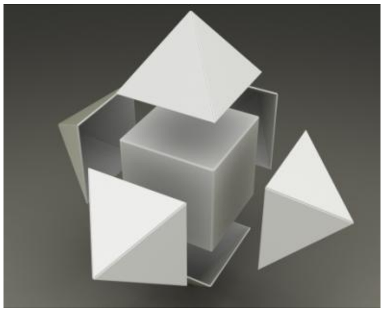


Figure 11: *Exploded view.*

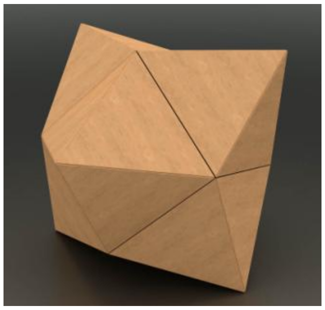


Figure 12: *Elevated Cube.*

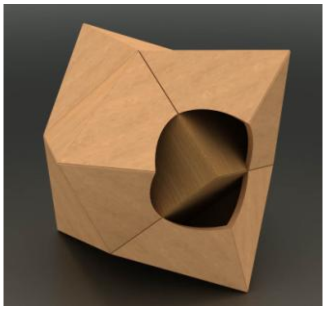


Figure 13: *First loop.*

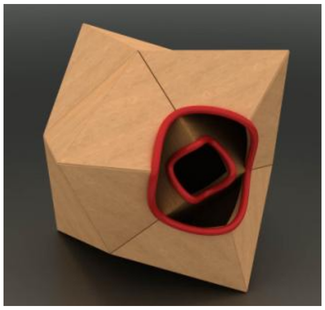


Figure 14: *Two separated loops.*

Looking at the vertex figure around the one of the vertices of the original cube we see again two separated loops, when we follow the description of Luca Pacioli. And these loops can be described as 333333 plus 444. The steps are shown in Figure 12, 13 and 14. Also in this case we have the situation that four faces come together at the edges that meet in these vertices. So also here we can choose an alternative order. Instead of just following the outside shell or the inside shell we start at one of the faces of the cube and then step over to the adjacent triangle face and after a second triangular face we step back on a face of the cube again. The total order of the faces of our walk will be 433433433 (Figure 15).

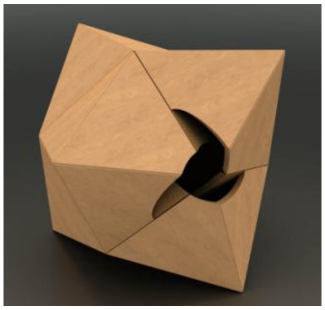


Figure 15: *Alternative path.*

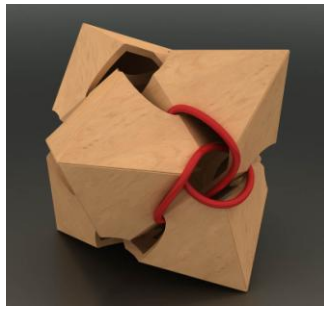


Figure 16: *Trefoil knot.*

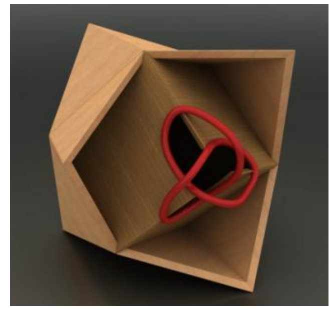


Figure 17: *Single surface.*

But that means that we now have one single loop that makes two tours around the vertex. We can modify the shape of the loop so that, when we use it as a cutting line in the total construction, all edges are connecting lines between just two faces (Figure 16). The total shape of the loops then will show the trefoil knot (Figure 17). By definition the "elevation element" for the elevated cube is the object shown in Figure 18. For the real physical construction we cut away some parts following the knot-line. And now the final model can be build with 6 of these elements. The end result is a single surface forming a double covering around the cube.

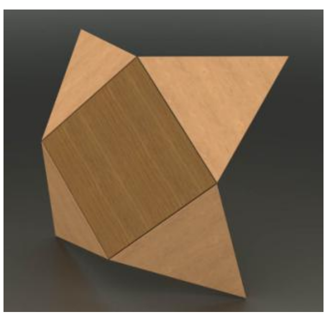


Figure 18: *Elevation element.*

Figure 19: *Construction element.*

Figure 20: *"Double Cube".*

3. General Observations

3.1. Holes and Compounds. Pacioli's original description refers to a cube within a shell of triangles. But if we take the alternative order of the faces around one of the cube vertices, we obtain instead a single intertwined double layer structure. And there is also another difference: the path around the cube vertices in the elevation gave us knot-shaped cutting lines for the holes in the construction instead of the simple loop-shape holes we got around the vertices of the octahedron in the elevated octahedron. In Figure 21 and Figure 22 rounded models, made on a 3D printer, of both the elevated octahedron and the elevated cube are shown. We can see that both objects are intertwined and double layered. The object of Figure 21 is an entwinement of two different parts, the object in Figure 22 is one single object. The shape of the holes in the elevated octahedron are simple loops, in the elevated cube we can recognize the trefoil knots.

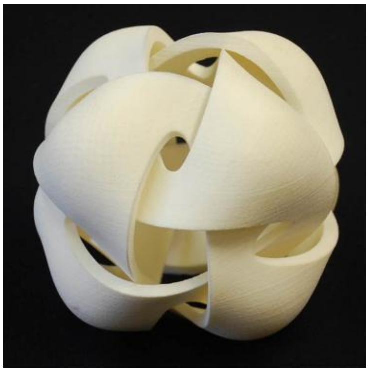


Figure 21: *Elevated Octahedron*

Figure 22: *Elevated Cube.*

3.2. Questions. A few questions may arise: If we can make a knot-shaped path around a vertex of an elevated polyhedron, do we always get one single double layered entwined object? When we want to make a single double layered entwined object, is it only possible with knot-shaped holes? The answer to second questions is "No", just because we can show a few counter examples as are shown in Figure 23 and Figure 24. But both these examples are built like a torus. And a torus has a higher genus then the sphere, the basic shape of the polyhedra that Pacioli used for the elevations. So the question might be: How can we understand this? In the elevated "spheres" the connectivity is determined by what happens locally around each original vertex. In the torus structures, the connection that turns these objects into single surfaces happens globally when traveling around the toroidal loop.

Figure 23: *Double Skin Torus.*

Figure 24: *Trefoil Torus.*

4. Connected Holes

4.1. Interwoven Layers. To get a better understanding of the connection between the shape of the holes and the number of parts in the construction, we start with structure of two interwoven surfaces as shown in Figure 25. All the holes in the construction are simple loops. Bending the construction will not change the properties of the holes (Figure 26). We can bend it so far that the left edges get connected to the right edges. But in this situation the right edge of one of the surfaces is connected to the left edge of the other surface and vice versa. So it is not two surfaces anymore, but one single interwoven surface, which is topologically equivalent to a double looped cylinder with a lot of holes (Figure 27).

Figure 25: *Elevation element.*

Figure 26: *Bending.*

. **Figure 27**: *Cylinder*.

4.2. Knots. When forming the cylinder, the nature of the many holes has not changed. Do we have a single double layer entwined object without knot-shaped holes? The answer is No, because at the moment the construction closed to a cylinder two extra holes were generated at the two ends of the cylinder. For the doubly covered cylinder in Figure 27, the edges of these two holes would form trefoil knots. Whenever we have an odd number of ridges on the cylinder, these edges of the end holes form single torus knots and the resulting shape is a single surface. For an even number of ridges, we will get two intertwined torus knots and the result would be two separate, intertwined cylinder surfaces. And these holes can be seen as trefoil knots (Figure 28).

Figure 28: *Knot.*

Figure 29: *Bending again.*

Figure 30: *Torus.*

4.3. Torus. In the simplest case the cylinder has just 3 ridges and the holes at the two ends are bordered by simple trefoil knots. Several such cylinder elements could be stacked on top of one another, since the upper and the lower edge curve have the same shape and can thus readily be fused together. In the connection two knot-shaped holes will disappear, they will be transformed to single loops again, and so the total number of knot-shaped holes will be two again. There is another way to connect the upper side to the underside. We don't need more copies of the structure, we can just bend again as in Figure 29. But then, when the upper side gets connected to the underside all knot-shaped holes disappear. We now have the situation that we have a single interwoven double layer object, but we needed to make it like a torus to get this result (Figure 30) which is similar to Figure 24.

5. Connecting the Knots

5.1. Reducing the number of knots. We have seen that we can reduce the number of knots by connecting them. Let us go back to the Pacioli's elevation of the cube. We will take the version with the alternative order of the faces around the cube vertices, the "double cube" of Figure 20.

Figure 31: *Connected elevated Cubes.*

Figure 32: *RP-model.*

Figure 33: *Connected Cubes rounded.*

We can now connect two of these double cubes by removing one of the pyramids plus the underlying face of both of the elevated cubes (Figure 31). Around the new vertices where two double cubes get connected, the edge of the holes are now changed into sets of pairs of intertwined loops, since the valance of the connecting vertices between two plain cubes is now 4. To make this more visible we can round off all the pyramids. In Figure 32 this is done for the single elevated cube and we can see the knot-shape holes at the vertices In Figure 33, the rounded of version of the connected elevated cubes, and now we see the linked loop-shape holes in the middle.

5.2. Linked Cubes. The rounded versions of the elevated polyhedra are in fact polyhedra in which each face is doubled. The single rounded version of the elevated cube has 2 x 6 "square" faces. The connected elevated cube model has 2 x 10 "square" faces. In the vertices in the middle of the object in Figure 33 we see 2 x 4 faces coming together and in the other vertices 2 x 3 faces are coming together. The odd number 3 leads to a knot-shaped hole, whereas the even number 4 leads to the two entwined simple loop-shaped holes.

5.3. Linked Squares. Looking at the connection of 4 elevated cubes (Figure 34), we can think of other applying the "doubling transformation" on other patterns with squares.

Figure 34: *4 Connected Cube.*

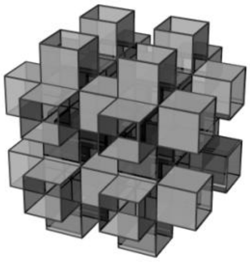
Figure 35: *Square pattern.*

Figure 36: *Elevated Squares.*

The cube is the regular polyhedron build with squares. In the plane we have the regular tiling with squares (Figure 35). Also here we can "double" each of the squares to make a construction similar to the idea of Elevation. The elevation of this pattern, or the "doubled square pattern" is shown in Figure 36. As we can see, this structure falls apart into two entwined layers, so it is a compound.

6. Coxeter's infinite Polyhedra

6.1. Elevation of the Coxeter's infinite polyhedron 444444. Beside the cube and the regular square tiling there is one other regular structure build with squares that can be studied in this context. This is Coxeter's infinite polyhedron 444444 (Figure 37), in which 6 squares meet at each vertex.



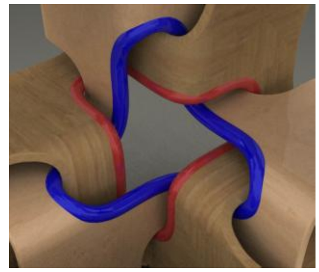
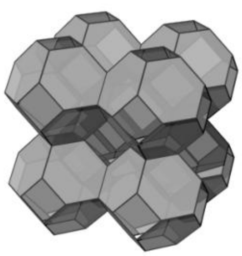


Figure 37: *Coxeter 444444.* **Figure 38**: *Elevated Coxeter 444444.* **Figure 39**: *Around the Vertex.*

It turns out to be possible to make a "double" version of this infinite polyhedron as can be seen in Figure 38. The situation around each of the vertices can be described as the entwinement of two simple loops (Figure 39). I think we can see the structure in Figure 38 as the elevation of Coxeters polyhedron 444444. And this elevation is a compound, two separate structures, entwined in space.

6.2. Elevation or "Doubling the Faces". The technique we applied to create the "double" version of Coxeter's polyhedron 444444 can be seen as a generalization of the Elevation operation. In Figure 41 the "double" version of Coxeter's polyhedron 6666 (Figure 40) is shown.



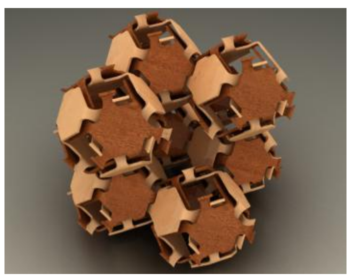


Figure 40: *Coxeter 6666.* **Figure 41**: *Elevated Coxeter 6666.*

6.3. Conclusion With the introduction of Elevation by Luca Pacioli and Leonardo da Vinci the interesting field of double layer structures has been opened. This paper is just meant to be an introduction to some more creative possibilities of this "new" concept, in addition to two former papers about this subject [4] and [5]. And still there is a lot to be investigated.

References

[1] Luca Pacioli – Leonardo da Vinci, *La Divina Proportione,* 1509, Ed. Akal, S.A., Madrid, 1991
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