# A Mathematical Approach to Obtain Isoperimetric Shapes for D-Form Construction

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## Abstract

Physical D-forms are obtained by joining the boundaries of two flat shapes with the same perimeter. To ensure both possess identical perimeters, one usually joins two pieces with identical shape (e.g., two congruent ellipses. Yet, using two different shapes increases the design possibilities significantly. In this paper, we present a simple mathematical approach to obtain two isoperimetric shapes for the physical construction of more general D-form surfaces. Using this approach, we have developed plug-ins for Adobe® Illustrator and Inkscape, which we tested in two freshman-level architecture studios. All of the freshman students were able to construct interesting physical D-form surfaces and then, using these physical form surfaces as casts, were able to build interesting concrete sculptures.

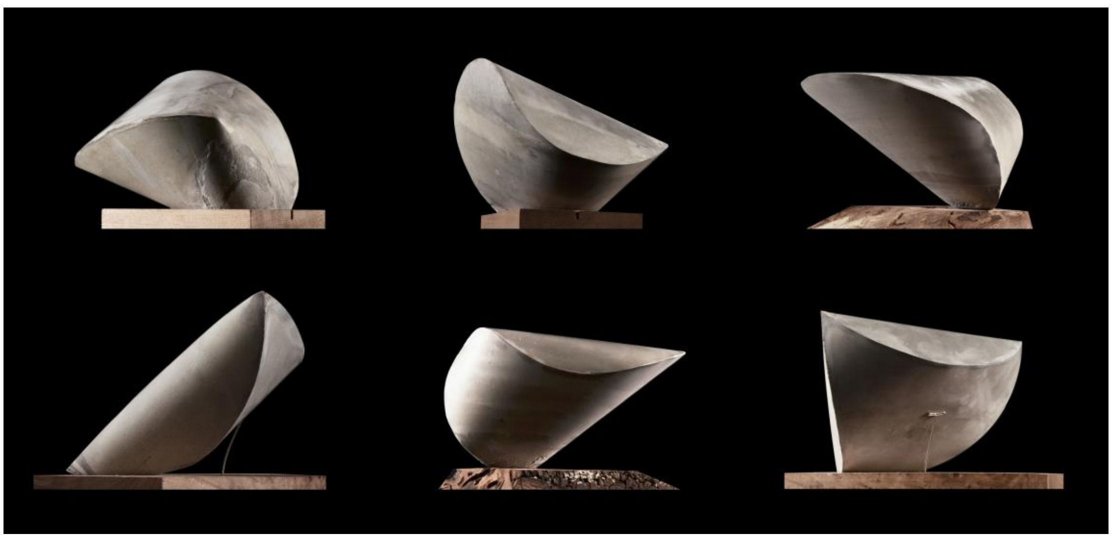


**Figure 1**: *Examples of concrete D-forms that are obtained using our methods. Freshman Design Studio, (Fall 2015), Texas A&M.*

## Introduction and Motivation

Developable surfaces [1] are particularly interesting for architectural design [2] as they have the property that they can be constructed from planar materials, like sheet metal, without extensional deformation. By physically constructing developable surfaces, it is possible to find novel forms [3, 4, 5]. A very interesting method to build developable surfaces, called D-forms, was invented in early 2000s by the London designer Tony Wills [6]. This relatively new method was first formally introduced to the art and math research community by John Sharp in 2005 [7, 8]. D-form surfaces are created by joining the edges of two flat surfaces that possess perimeters of identical length [6, 7]. Importantly, Willis and Sharp demonstrated that most flat surfaces can produce a large variety of 3D shapes: altering the initial point along which the two edges are joined changes the shape and volume of the resulting solid. Since then, there has been a growing interest using versions of this method to create complicated surfaces [9, 10].

The simple elegance of D-forms, especially in creating a wide variety of aesthetically appealing shapes, makes them ideal for use in introduction-level architecture studios. But creating two different shapes whilst ensuring that they have the same perimeter is not straightforward in general. No prescription, thus far established, enables one to obtain two different shapes with the same perimeter easily and flexibly — unless they are geometric shapes whose perimeters are given by well-known analytical formulas. Smooth geometric shapes for which such formulas exist are, unfortunately, fairly rare. Even for ellipses, no exact analytical formula exists to compute their perimeters [11]. For this reason most people create D-forms with two identical flat shapes. This paper presents a mathematical approach to obtain isoperimetric shapes starting from two boundary curves with different lengths.

Based on this approach, we have developed a software application called *D-former* to aid users in the creation of shapes with perimeters suitable for the created of D-forms. We implemented D-former as a plugin for Inkscape, an open source vector graphics software, and Adobe® Illustrator®, a commercial vector graphics software suite. We have also provided a feature to add “connectors” that is particularly useful to users interested in physical construction of the form, for example, by assembling pieces produced by a laser-cutter.

## Mathematical Foundations

This section presents our new mathematical framework that provides an explanation for the aesthetic appeal of D-form surfaces and shows that we can always obtain desired perimeters by uniformly scaling the original curve.

**The Visual Aesthetic of D-forms:** We recently noticed that D-forms are really the result of a continuous version of angular defect. Suppose that the two curves bounding the surfaces to be joined are denoted $C_0(t)$ and $C_1(t)$. In D-forms we observe that the angle defect is distributed along the curve. Let the angular change in any of the connection positions of the two be denoted by $\theta_0(t)$ and $\theta_1(t)$ — In Discrete Differential Geometry angular change is also called turning angles or Discrete Gauss map (see [14]) —then the change in angle defect at that point is defined, based on these two angles, as

$$\mu(t) = 2\pi - \theta_0(t) - \theta_1(t). \qquad (1)$$

We note that this is related to winding number and 2D Gaussian curvature. From Differential Geometry we know that for single simple closed curves, parameterized by a parameter t between 0 and 1, the following equation always holds [14].

$$\int_0^1 (\pi - \theta_0(t))dt = 2\pi. \qquad (2)$$

This equation is really the 2D version of the Gauss-Bonnet theorem for closed simple curves. If we take the integral of Equation 1, we obtain a result that is consistent with 3D Gauss-Bonnet theorem [12] as follows:

$$\int_0^1 (2\pi - \theta_0(t) - \theta_1(t))\, dt = 4\pi. \qquad (3)$$

Note that here $4\pi$ is exactly the Euler Characteristic of a genus zero surface (which is 2) times $2\pi$, as predicted by the Gauss-Bonnet theorem. In other words, in a D-form, Gaussian curvature is zero everywhere except along the 3D curve that is created by joining the two (originally planar) curves. By changing the starting point, we simply change the distribution of Gaussian curvature on the 3D curve. Since along a smooth curve the value of the Gaussian Curvature changes smoothly, the resulting D-form's surface provides a smooth curvature change in 3D. Most likely the visual appeal of D-form surfaces comes from this property. As noted, we may alter the distribution of local curvature simply by changing the first connection point.

**Uniform Scaling:** The key idea behind our approach for equalizing perimeters is that uniform scaling of any shape changes its perimeter by exactly the scale amount. Although it is difficult to supply an exact formula for the perimeter of a shape, it is always possible to estimate its perimeter by using a reasonably dense piecewise linear approximation of the boundary of the shapes, itself a closed curve.

Let a closed curve $p = C(t)$ be defined by a parametric function, such as a Catmul-Rom or B-Spline polynomial [13], with a periodic parameter t, where $C(t)$ is a function from [0, 1] to $\mathbb{R}^2$ with $p = (x, y)$. Regardless of the underlying parametric function, curves are drawn in a vector drawing software packages as piecewise linear curve passing through a set of n points such as $(p_0, p_{1,} p_i, p_{i+1,} p_{n-1})$. From this approximation, the perimeter of an approximated shape can be computed as follows:

$$A = \sum_{i=0}^{n-1} |p_{i+1} - p_i|$$

where $|.|$ is an $L_2$ norm and + in the subscript is a summation modulo n. Now, we apply a uniform scaling operation via a scaling parameter s around a given pivot point $p_c$, so that every point is transformed by the following equation as

$$\acute{p} = s\,(p - p_c) + p_c$$

where $\acute{p}$ is the transformed position. Note that the distance between two points is independent of the choice of pivot point as follows,

$$\acute{p}_{i+1} - \acute{p}_i = s\,(p_{i+1} - p_c) + p_c - (s\,(p_i - p_c) + p_c) = s(p_{i+1} - p_i)$$

In other words, if s is positive real, then $|\acute{p}_{i+1} - \acute{p}_i| = s\,|p_{i+1} - p_i|$. This means that the new perimeter is the scaled version of old perimeter as follows:

$$\acute{A} = \sum_{i=0}^{n-1} |\acute{p}_{i+1} - \acute{p}_i| = \sum_{i=0}^{n-1} s|p_{i+1} - p_i| = s\sum_{i=0}^{n-1} |p_{i+1} - p_i| = sA$$

Thus, if we know current perimeter of a given shape, we can always obtain a new shape with the desired perimeter by uniformly scaling.

**Methodology:** Using uniform scaling, creation of two curves with equal perimeters that can be stitched together easily requires the following steps:

1. Draw two closed curves using a parametric function such as B-Spline or Catmull-Rom [13].
2. Normalize the shape perimeters by scaling both curves to a given perimeter value.
3. Add connectors and holes to join curves easily once they are printed or cut.
4. Using a laser cutter, obtain the two curves with their connectors.

5. Construct D-form surfaces using these curves.
6. Create concrete D-Forms using D-Form surfaces as casts (See Figure 1).

This whole process could be implemented with stand-alone software, but with the disadvantage that providing powerful user-interface tools is involved, not to mention requiring a user to become familiar with the software. Therefore, instead of implementing a new drawing program from scratch, we extended two existing products which already have sophisticated user interfaces.

## Implementation

We build D-formers as a plug-in on top of two existing and widely used 2D vector drawing software programs: Adobe Illustrator and Inkscape. Figure 2 shows the block diagram of our D-former plug-in. The blocks in *blue* are the scripts that are part of our solution. These scripts are: Normalize-Perimeters, Add-Stitching, Add-Leaves, and Add-Notches. The Normalize-Perimeter script ensures that the D-form conditions are met. The other three "Add" scripts provide support for users in the later physical construction of D-form surfaces. Purple blocks represent internal components, or functions, that the scripts rely on to accomplish their individual task. The bidirectional arrows are used to indicate a dependency (the flow of input and output). The arrow between the top and the middle layer show that scripts will need input from the software and will eventually modify the desired document in the two applications.

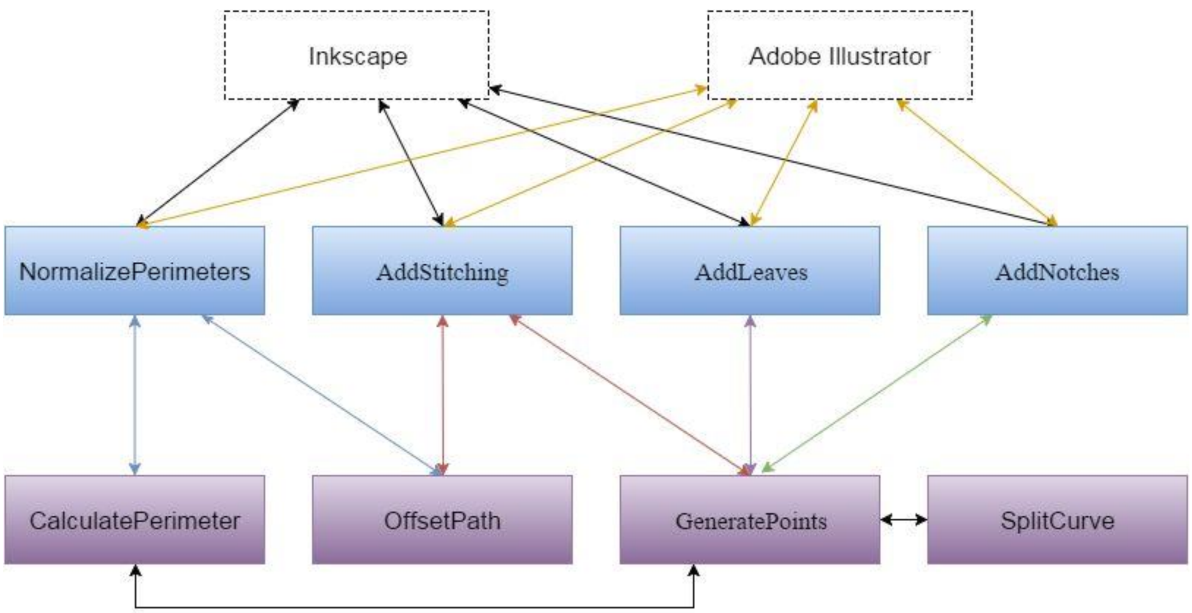
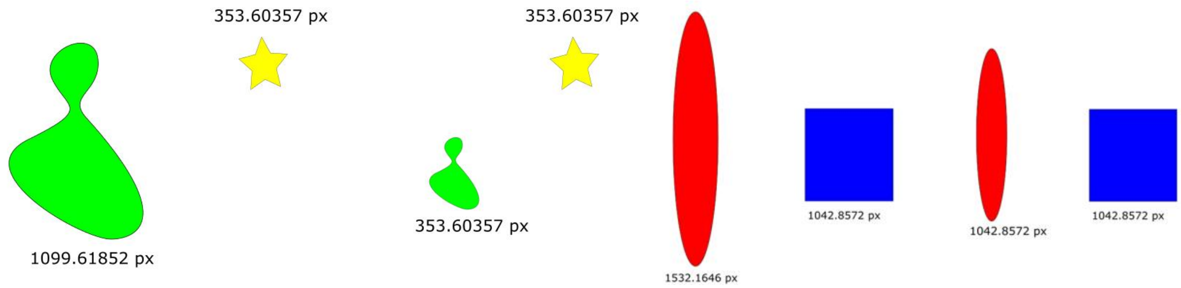



***Figure 2****: Block Diagram of our D-Former Plug-Ins.*

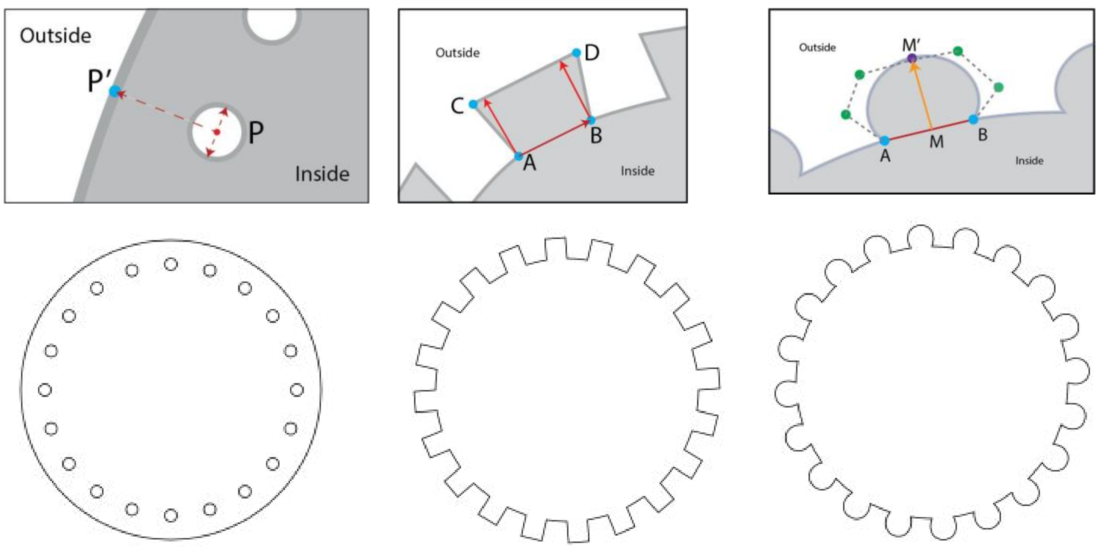
Figure 3 shows screenshots from the program that demonstrates the effect of Normalize-Perimeter operation on two random shapes.




*Original Input Shapes* *Isoperimetric Shapes Produced* *Original Input Shapes* *Isoperimetric Shapes Produced*

***Figure 3****: Examples showing how isoperimetric shapes are obtained.*

Add operations insert equally spaced holes, notches, and leaves. Figure 4 shows the difference between these three forms of connector.




*Holes (Stitching)* *Notches* *Leaves*

***Figure 4****: The difference between stitching, notches and leaves.*

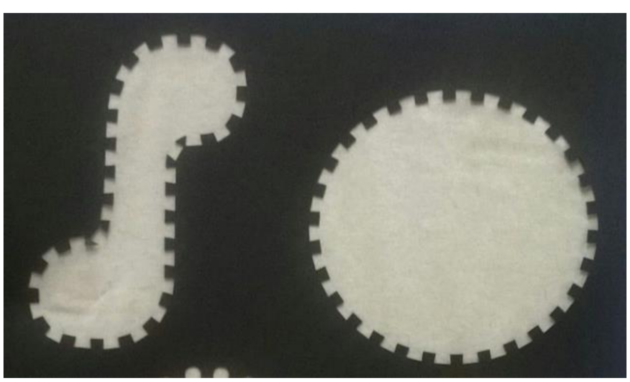


***Figure 5****: Two laser cut planar shapes with notches.*

Figure 5 shows two laser cut pieces with notches. These plug-ins are now used in freshman design studio to obtain complicated D-forms such as the ones shown in Figure 1. Though it was not difficult to physically construct D-forms out of relatively simple planar shapes, it was challenging for freshman students to create complicated D-forms without calculating perimeter lengths. Without the D-formers plug-in, nearly all of them were unable to calculate or even estimate the perimeter of an irregular shape made from curved lines. With this plug-in, detecting the perimeter relationships of pieces with unusual and sophisticated boundaries became quite easy for students.

*Figure 6: A concrete D-form sculpture made out of an organic shape and a circle (Designed by Brittney Martinez). Freshman Design Studio (2015), Texas A&M.*

In the freshman studio, building a D-form that embodied the principles of developable surfaces began with plane geometry and progressed to the three-dimensional. Depending on the complexity and similarity of the two planar shapes, a variety of results were achieved. By moving from the simple to the more complex, and making a range of volumes from a single pair of flat shapes, the D-former plugin assisted students in designing new forms, in a wide variety of scales, and within a multiplicity of contexts. For many, it was difficult to believe that complex forms, with their various beautiful and pronounced curves and twists, were generated from only two pieces of planar material (see Figure 6).

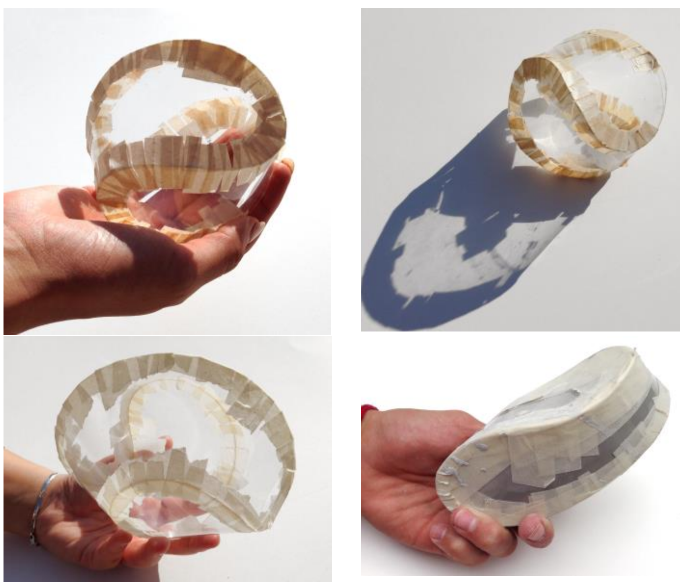


***Figure** 7: Examples of D-form surfaces used for casting.*

Working on D-forms provided beginner design students with the opportunity to tinker and cross disciplinary boundaries during their first year of undergraduate study. This helped them engage with new modalities of research and practice, spanning fields of mathematics, engineering, computer science, art, and embracing design as an overarching umbrella. D-form explorations, as the first studio assignment, brought the power of geometry to the attention of these freshman students. Here, besides disseminating new knowledge about geometry, constructability, and materiality, D-forms helped students tap into and use their prior mathematical knowledge as they designed and made their D-forms. The knowledge of geometry and the dexterity the students possessed at the beginning of the semester grew steadily throughout the rest of the semester, informing their other assignments positively.

## Casting Process

To best realize D-form designs of the students, sculptures were cast in concrete. By speculating on the gravitational and hydrostatic forces of concrete objects in a liquid state, the last phase of this assignment honed students' ability to look closely at casting techniques (Figures 1 and 6). They were asked to design bases for their concrete pieces that retained a sense of the weight and mass inherent in the material.
Students were invited to create the most aesthetically pleasing D-form sculptures possible; their designs were developed from a combination of concepts drawn from mathematics and various formal approaches. In these sculptures, the relationships among craftsmanship, originality, creativity, and spatial characteristics were also important considerations (See Figure 7).

***Figure 8****: Casting process.*

For many, it was difficult to believe that all of the sculptures, with their various beautiful and pronounced curves and twists, were generated from only two pieces of planar material. By dealing with a heavy concrete mix that gradually changed from liquid to solid and utilizing formworks made from plastic sheets with very limited strength, elasticity, and stretch-ability, almost all of the sculptures evolved into rock-hard D-forms with no major wrinkles on their surfaces or other forms of defect such as shrinkage cracks or blistering (See Figure 8).

## Conclusion and Future Work

We hope in the future to construct one or two D-forms on a larger scale with the help of the software we have developed. We are now exploring improvements to the software, so that a user can design D-forms and directly visualize the final form in the software before engaging in physical construction. Figure 9 shows our current implementation of a visualizer for D-forms: given two input shapes, it uses an optimization formulation to determine the final geometry. After it is designed, the desired D-form can be unfolded within the software so that it can then be laser cut.

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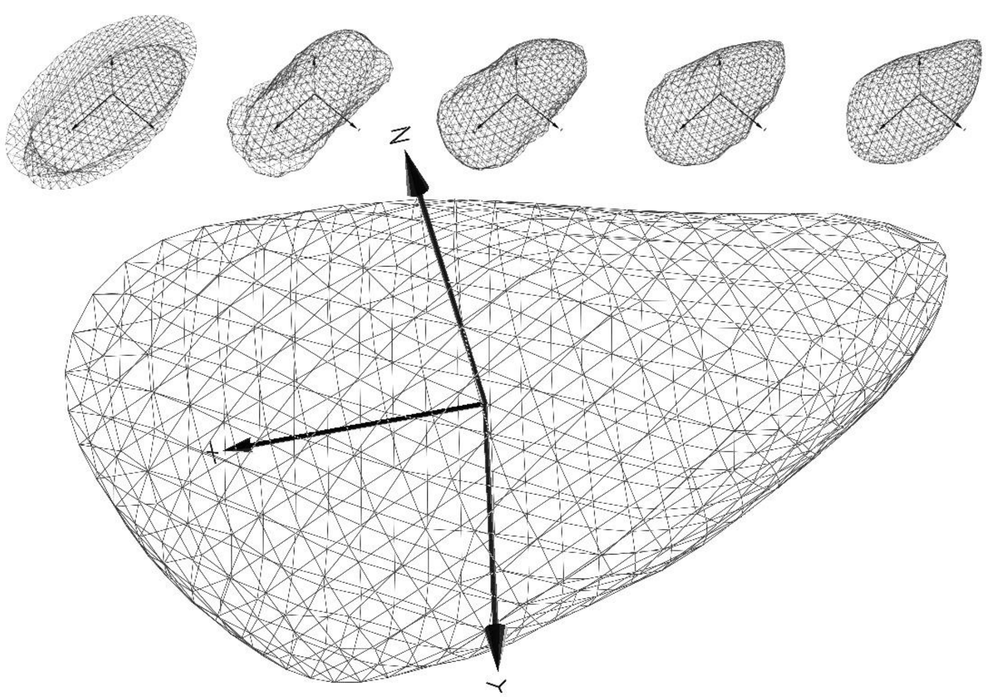


***Figure 9**: A visualization of mesh relaxation rendering of a D-Form made from two ellipses, one rotated through 54° and stitched to the other. The five inset images show stages of the iterative Levenberg-Marquardt solver minimizing a least-squares optimization problem that penalizes gaps between stitches and non-planarity in the pieces.*