Hex Rosa

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Abstract

This paper describes a system of rhombic tilings with n-fold rotational symmetry for all $n \geq 3$. A preference is given to the presentation of odd values. In addition to one centre of global rotational symmetry this system contains infinitely many relatively small evenly distributed circular patches with their own centres of n-fold local rotational symmetry. This system uses specific hexagonal modules and certain properties of them are also described.

Roses from Penrose

The famous Penrose tiling [1] was the original inspiration for the system of rhombic tilings described in this paper. Nevertheless the properties of this system are much simpler than those of the Penrose tiling. Unlike the Penrose tilings done with two rhombuses with the specific matching rules, which guarantee them being aperiodic [2], this system uses rhombuses with no matching rules; thus no tiling described here is aperiodic. However, except for values n = 3, 4, and 6 all the tilings described are nonperiodic.

Figure 1 shows some patches for the first few values of n. As these patches greatly resemble flowers with their petals I call them *roses*. The term *rosette* is also used for similar types of rhombic compositions but it is also used of a patch, which has exactly the same rhombuses inside an identic regular perimeter but in any possible order [3]. Thus I use here the word rose to describe patches seen in Fig. 1 with both rotational and reflection symmetry. The 5-fold symmetric roses are easily seen in all rhombic Penrose tilings, see, for example, Fig. 10.3.20, (pp. 544–545) in Grünbaum and Shephard [2].

Rose patterns in Fig. 1 can be seen as two-dimensional projections of three-dimensional polar zonohedrons [4]. It is even difficult *not* to see some illusory three-dimensionality in these patterns, especially as the leftmost pattern in Fig. 1 is a typical way of depicting a cube. These patches emerge also in some two-dimensional projections taken from even higher n-dimensional hyper-cubic lattices [5]. The results described in this paper are only a summary of about 25 years of intermittent studies based on few simple ideas and observations. More serious theoretical results have later developed from the tilings described here and they have also been published recently [6].

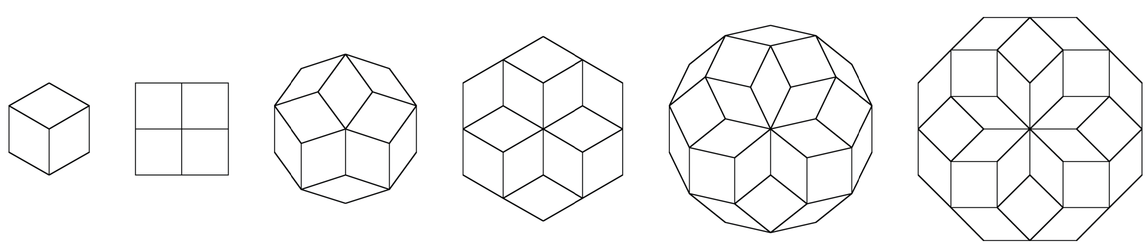

Figure 1: *The roses for n = 3, 4, 5, 6, 7, 8.*

Hexagonal Modules

In Figure 2 one can see a multitude of roses repeating seemingly *ad infinitum* in all directions. Actually this is the case and corresponding tilings can be constructed for all $n \geq 3$. This is achieved using a certain repeating hexagonal module, shown with grey lines in Fig. 2. I call this hexagonal module as *delta hexagon*.

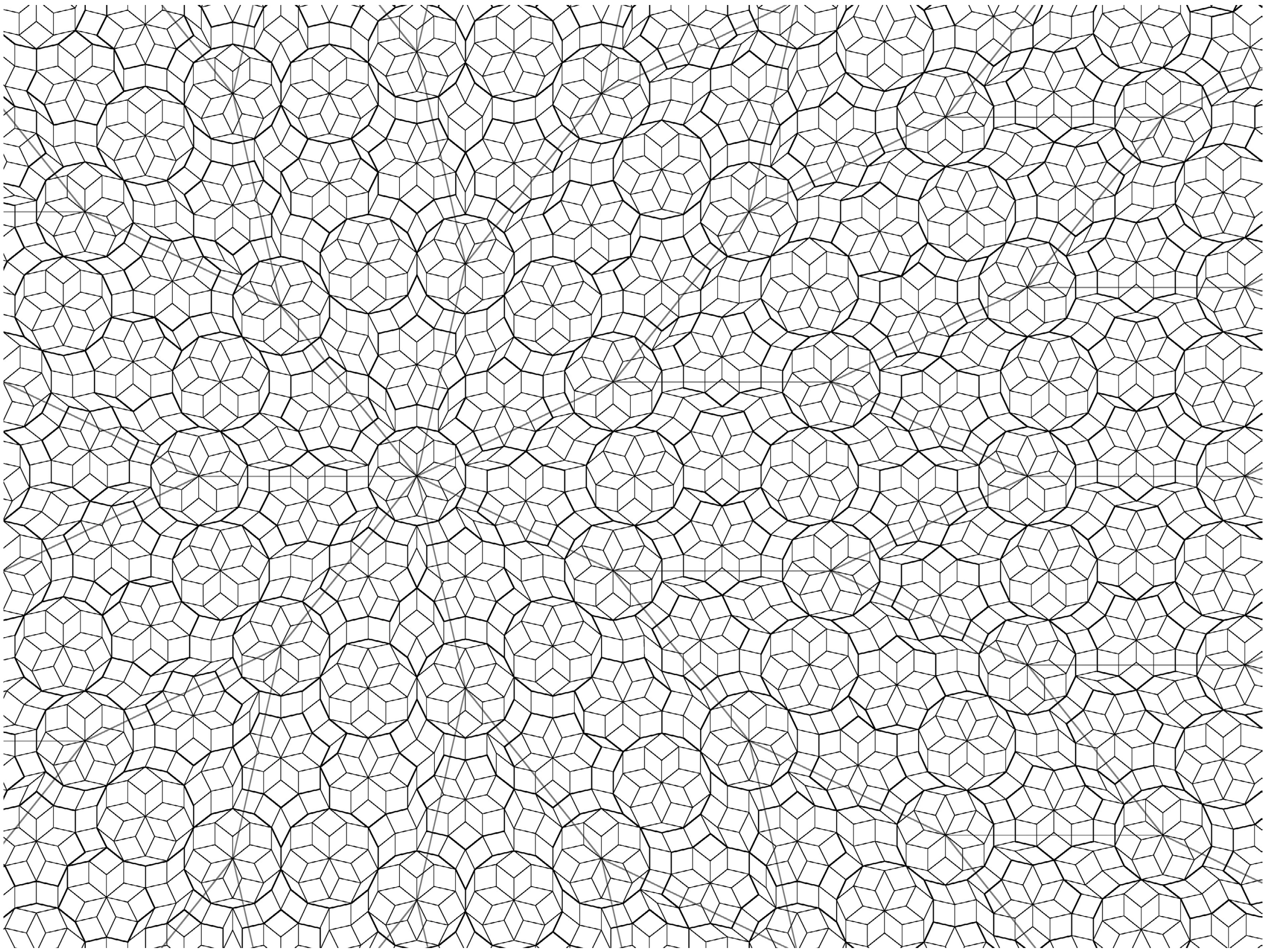

Figure 2: *The Hex Rosa tiling for n = 7 with its hexagonal modules also shown.*

The delta hexagon has all its sides of same length and all the opposite sides parallel. One main property of the delta hexagon is defined by an integer k, which divides 360° in k equal parts. The interior angles of the delta hexagon are $360°/\ k$ in two opposite corners and $180°((k–1)/\ k)$ in four other corners, the total sum of interior angles being always 720°.

In May 1999 I made the simple observation that a plane can be tiled with these hexagons for all $k \geq 2$ in the following manner. First we place k delta hexagons tightly around a point in such a manner that all these hexagons share sides with their nearest neighbours. We continue the arrangement by placing k congruent hexagons between each two previous hexagons, between them we place $2k$ hexagons, between them we place again $2k$ hexagons, and so on with $3k$, $4k$, $5k$, etc. hexagons. This produces a pattern made of delta hexagons seen in Figure 3. Compare Figs. 2 and 3 for $n = 7$, note that these images are drawn in different scales.

Another more simple way of tiling the plane with hexagons is the periodic "chicken-wire" pattern. In Fig. 3 we see how there are 14 (or generally $2k$) areas or "wedges", which actually consists of the "chicken-wire" pattern. The borders of neighbouring "wedges" have a zigzag form depicted with darker lines in Fig. 3. Delta hexagons are called such because triangularly growing wedges made of them slightly resemble a river delta in their appearances, see Fig. 3.

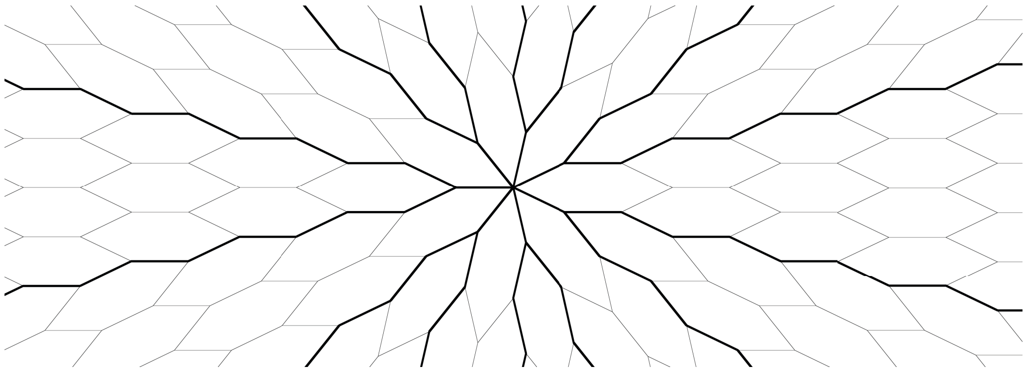

Figure 3: *The fourteen delta hexagon wedges for k = 7. Half of them meet in the centre.*

Only k wedges meet in the centre and these wedges separate another k wedges. See Figures 4 and 5 for four corresponding delta hexagon wedges for $k = 2$ (left) and six wedges for $k = 3$ (right). The hexagons are always connected only edge-to-edge. Note that for $k = 2$ the delta hexagon with its six vertices of 90°, 90°, 180°, 90°, 90°, 180° is also a proper rectangle with four vertices of 90°.

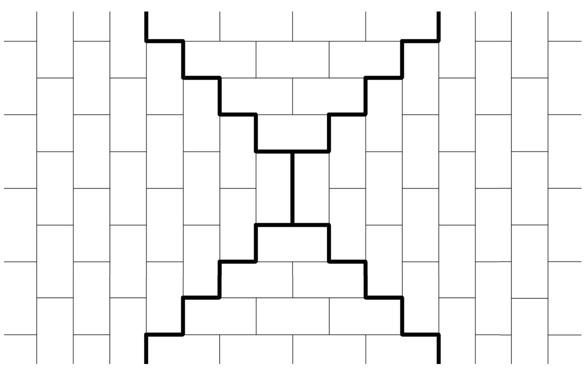

Figure 4: *The four delta hexagon wedges for k = 2*

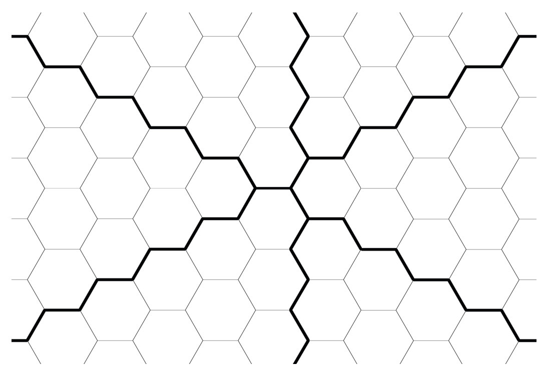

Figure 5: *The six delta hexagon wedges for k = 3*

The nonperiodic k-fold rotationally symmetric delta hexagon pattern and the more typical "chicken wire pattern" are by far not the only possible tilings with delta hexagons; actually there are infinite many non-congruent tilings possible with delta hexagons but I will not go into them in this paper.

Filling the Modules

In the end the delta hexagons are treated here only as provisional modules to obtain rhombic tilings with a fairly high concentration of locally n-fold symmetric roses for all $n \geq 3$. As this system uses hexagons and rose patterns it is called *Hex Rosa*. Such a tiling is obtained here by placing a rose at every vertex of a delta hexagon. This guarantees a tiling with an evenly and nonperiodic distribution of infinite number of locally n-fold rotationally symmetric patches plus one centre of globally n-fold rotational symmetry.

A Hex Rosa tiling can be constructed for all $n \geq 3$. In Fig. 2 we see there are actually two types of roses –convex and concave– in a Hex Rosa tiling. First there are the convex or "closed" roses, which have their perimeter in the shape of a regular polygon, as seen in Fig. 1. Secondly there are the concave or "open" roses in between the convex ones. Peeling off the outermost "petals" from the closed roses gives the corresponding open roses for all $n \geq 5$. For $n = 3$ and 4 the open roses are mere thin sticks with no area. In addition to these new concave roses and convex roses positioned in the vertices there are two more convex roses inside the hexagon, see Figures 7 and 8.

A distinction between odd and even values of n needs to be done. The perimeters of the roses are regular $2n$-gons of (with edges of one unit) for odd values and regular n-gons (with edges of two units) for even values. The rotational symmetry of a rose is n-fold for all n, odd and even, see Fig. 1. Every n belongs to one of the following three cases, which characterize the Hex Rosa tilings:

1) If n is odd there are $(n–1)/2$ different prototiles, or elementary rhombuses: $(1, n–1)$, $(2, n–2)$, … $((n–1)/2, (n+1)/2)$, written as multiples of the angle $180°/n$.

2) If n is even and $n/2$ is odd there are $(n–2)/4$ different elementary rhombuses: $(1, (n–2)/2)$, $(2, (n–4)/2)$, … $((n–2)/4, (n+2)/4)$, written as multiples of the angle $360°/n$.

3) If n is even and $n/2$ is also even there are $n/4$ different elementary rhombuses: $(1, (n–2)/2)$, $(2, (n–4)/2)$, … $(n/4, n/4)$, written as multiples of the angle $360°/n$. Note that only in this third case there are squares among the rhombuses.

The differences between odd and even values are noticed already while defining the angles of the delta hexagons. For odd values of n we can identify simply $k = n$, thus the angles of the delta hexagon are $360°/n$ in two opposite vertices and $180°((n–1)/n)$ in four other vertices. For odd values there are n wedges meeting in the centre with another n wedges positioned tightly in between them, see Fig. 3.

For even values of n the identity $k = n$ would cause unsolvable problems later while tiling the delta hexagons with rhombuses. To avoid this problem we define $k = n/2$ for all even values of n. Thus for even n the angles of the delta hexagon are $720°/n$ in two opposite vertices and $180°((n–2)/n)$ in four other vertices. For even n there are $n/2$ wedges meeting in the centre with another $n/2$ wedges positioned tightly in between them, see Fig. 4.

The Rose Gardens

I call the delta hexagon, which contains roses in its corners and which is legitimately tiled with rhombuses a *rose garden*. Note that only a part of a rose in the corner is contained inside the specific hexagon and the rest of it belongs to neighbouring hexagons. As the sum of the interior angles of a hexagon is always two full circles the corners of one hexagon contain rhombuses for only two complete roses.

For $n = 3$ all roses can be oriented the same way (see Fig. 6). For $n > 3$ a specific orientation rule is needed to ensure that roses orientate correctly with the neighbouring rose gardens. The orientation rule is depicted in Figures 7 and 8: the long edge of the hexagon and one short line coming from the centre of a rose have to meet perpendicularly outside the hexagon and together they form a chain of six connected super-slim **L**-shapes defining the edges of the rose garden. When rose gardens are legitimately connected i.e. edge-to-edge, the roses at corners are cordially shared by all gardens meeting at the point in all possible combinations. I leave it to the reader to assure her self of this fact.

For clarity in the following three images (Figs. 6, 7, and 8) an excessive number of rhombuses are depicted outside the edges of the hexagons. Eventually the edges of a rose garden have to be "cut clean" to avoid overlapping with the rhombuses of the neighbouring hexagons, but to emphasize the number, position, and the shapes of the roses and rhombuses the sides of the gardens have been left "uncut" in the following images. To complete the Hex Rosa tiling for a plane it suffices to find a proper way to tile the finite interior of a rose garden as its delta hexagon shape allows the tilings of an infinite plane in a way seen in previous Figures 3, 4, and 5.

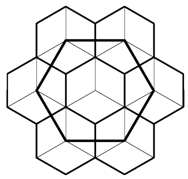

Figure 6: *The rose garden for n = 3.*

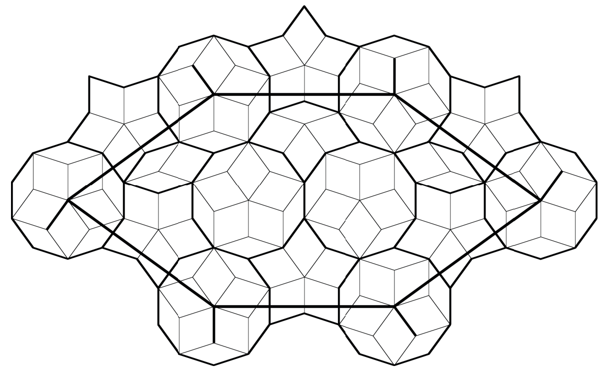

Figure 7: *The rose garden for n = 5.*

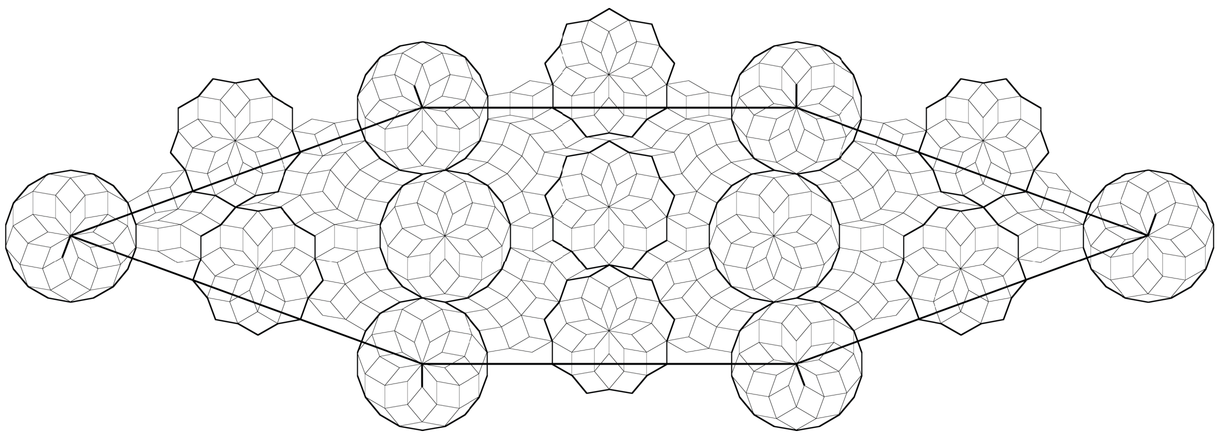

Figure 8: *The rose garden for n = 9.*

Paths to the General Solution

In the following I limit my presentation to the odd values of n. As n grows one must recognize certain rules and regularities in order to systematically construct new rose gardens for the larger values of n. Alternative solutions and variations naturally exist. I prefer a system where the rhombuses are arranged in such a way that straight lines from one rose centre to another rose centre will bisect all rhombuses along the line. I call these lines *paths*.

Figure 9 shows how these paths are located in a "general" rose garden with the value $n = 11$ used in this particular image. The convex or "closed" roses are numbered R1, … R8 whereas the concave or "open" roses are numbered R9, … R15. Roses R1, … R6 occupy the corners. Due to the lack of space convex rose R6 in the rightmost corner is cut out from the image as well are the two concave roses R14 and R15 from the same direction. I trust the reader is able to imagine the missing right hand part of Fig. 9.

The size of the hexagon in relation to the size of the rose is defined the following way. Lines R1-R2 and R1-R3 form an angle of 360°/n. There is a rose (R7) in between roses R2 and R3, both of which share one edge with R7. The distances R1-R2, R1-R3, and R1-R7 are equal as are the angles R2-R1-R7 and R7-R1-R3. Please compare Figs. 7, 8, and 9. There are different types of paths, marked with letters A, B, C, M, N, Aa, Bb, and Cc in Fig. 9. In addition to paths running from centre to centre there are paths, which either start or end at "tangent" points where some roses touch each other. These points are marked with a small circle and T(xxx) in Fig. 9. If we define the edge of the rhombuses to be one unit of length we have the following equations:

- The outer radius of the rose, i.e. the radius of the circumscribed circle $R(n) = 1/(2\sin(w))$,
- The length of the side of the rose garden hexagonal $S(n) = \cos(w)/2\sin^2(w)$, and
- The length of the paths $P(n) = 1/(4\sin^2(w))$, in all these three equations $(w) = 90°/n$.

$P(n)$ is the length of "full" paths A, B, C, M, and N. Path Aa is one rhombus shorter in length. Paths Cc and Bb are of full length, but exceptionally do not bisect their last rhombuses. There is an elegant connection between the length of the paths and the radii of the roses: $P = R^2$.

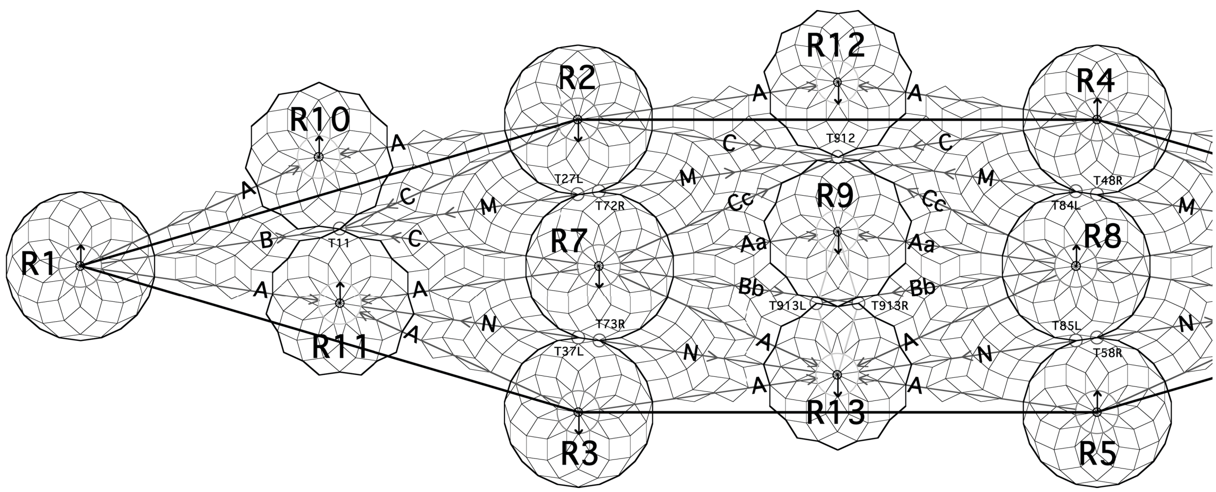

Figure 9: *The map of a general rose garden with its paths.*

The path A can be considered as the "archetype" for all these paths. The composition of paths B and C relate more closely to A than the composition of paths M and N do. The composition of a path can be coded with the bisected rhombuses.

In Fig. 9, for example, the path A from R1 to R10 reads as (0-4-8)-(0-2-4)-0-(6-2) and the path A from R2 to R10 reads as (2-6-10)-(0-2-4)-0-(6-2), please note the reading direction. The underlined values refer to the rhombuses inside the roses. In Fig. 9 the path B reads as (2-6-10)-(0-2-4-6)-(0-2), which has exactly the same rhombuses as the path A from R2 to R10, but in another order. All full paths are of the same length and they contain the same rhombuses with the possible exception of the underlined part, which runs inside a rose. Depending of its direction the radius $R(n)$ of every rose is either composed from rhombuses (0-4-8…) or (2-6-10…); compare, for example, rhombuses inside R1 along paths A and B.

Unfortunately I am not able to give extensive description of compositions and properties of the paths in this paper. I will just mention the following observations. It is rather straightforward how the complete tables expressed in rhombuses are formed for all paths A, B, C, M, N, Aa, Bb, and Cc for all odd *n*. For example for $n = 19$ the path A has the rhombuses (0-4-8-12-16)-(0-2-4-6-8-10-12)-(0-2-4-6-8)-(0-2-4)-(0)-(14-10-6-2) and the path M has the rhombuses (***0***)-(***4***-*2*-0)-(***8***-*6*-4-2-0)-(***12***-*10*-8-6-4-2-0)-(***16***-*14*-12-10-8-6-4-2-0), where the rhombuses can be re-arranged to obtain the composition of the path A. For example first rhombuses marked with bold italics (***0, 4, 8, 12, 16***) form the left underlined rose-part of the path A, i.e. (0-4-8-12-16) and the second rhombuses marked with normal italics (*2, 6, 10, 14*) form the mirror sequence of the right underlined rose-part of the path A, i.e. (14-10-6-2).

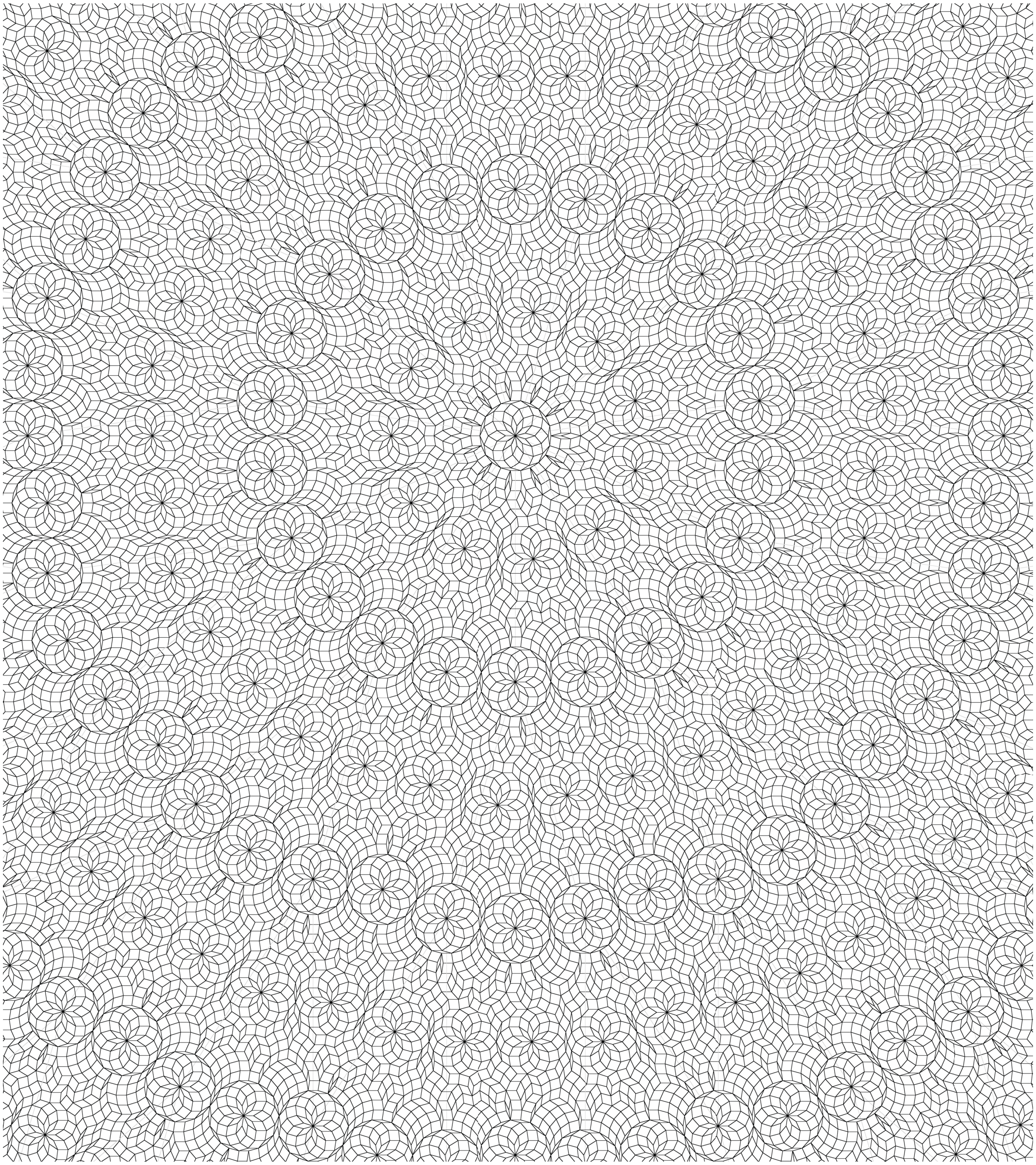

Figure 10: *The Hex Rosa tiling for n = 11.*

Conclusions

Paths paved with bisected rhombuses give a robust framework for tiling the complete rose garden for all n. In addition to these straight lines each type of path defines a "natural" tiling in its vicinity. Roses, paths and their surroundings combined cover the hexagonal rose garden almost completely. For larger n there are some small areas, which are left in between these systematically coverable three types of areas.

Nevertheless, I believe all such small areas can always be tiled ad hoc before further analysis reveals some complete rules also for them.

Beyond the Hex Rosa

In the end of 2011 the paths depicted in Fig. 9 gave me an idea about transforming the Hex Rosa tilings into substitution tilings. Like the actual tiling in the Hex Rosa consists of rhombuses some paths define also larger rhombuses with the same angles, for example paths A-A-B-C between R1 and R2, paths C-C-C-Cc between R2 and R7, and Cc-Cc-A-A between R7 and R8. This observation raised a natural question: if the unit rhombuses can be arranged to form such larger "super-rhombuses" would it not be possible to arrange these larger "super-rhombuses" to even larger "super-super-rhombuses"? Or, if this was possible, wouldn't it mean that the unit rhombuses can be seen as consisting of smaller "micro-rhombuses" and so on, ad infinitum, on both directions, zooming in and zooming out?

Such substitution was not achieved easily but eventually a viable solution was found. With the substitution process the hexagonal module was discarded and the Hex Rosa evolved into Sub Rosa [6]. It turned out that the Sub Rosa tilings are even linearly recurrent or quasiperiodic, i.e. for every finite patch there exists infinitely many identical copies of it within some finite distance, which is linearly dependent of the diameter of the selected patch. Together with the nonperiodic rotational symmetry for all n and the crystallographic restriction theorem, which forbids more than one global centre of rotational symmetry for values other than n = 2, 3, 4, or 6 there is a paradox-looking situation within such tilings. They must have one absolutely unique centre of global rotation and yet their quasiperiodicity guarantees that one is able to take as large (finite) area as wished also around this point and there exist an infinite number of congruent patterns, within some finite distances from each other.

The Penrose tiling was the first tiling with such a quasiperiodic property for a value (n = 5) other than the periodic ones n = 2, 3, 4, and 6. As the Sub Rosa proves [6] similar properties to exist for all $n \geq 2$, I can conclude by saying that the circle from my original inspiration, the Penrose tiling, has nicely closed.

References

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