The Conformal Hyperbolic Square and Its Ilk

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Abstract

We present explicit equations for three different mappings between the disk and the square. We then use these smooth and invertible mappings to convert the Poincaré disk into a square. In doing so, we come up with three square models of the hyperbolic plane. Although these hyperbolic square models probably have limited use in mathematics, we argue that they have artistic merit. In particular, we discuss their use for aesthetic visualization of infinite patterns within the confines of a square region.

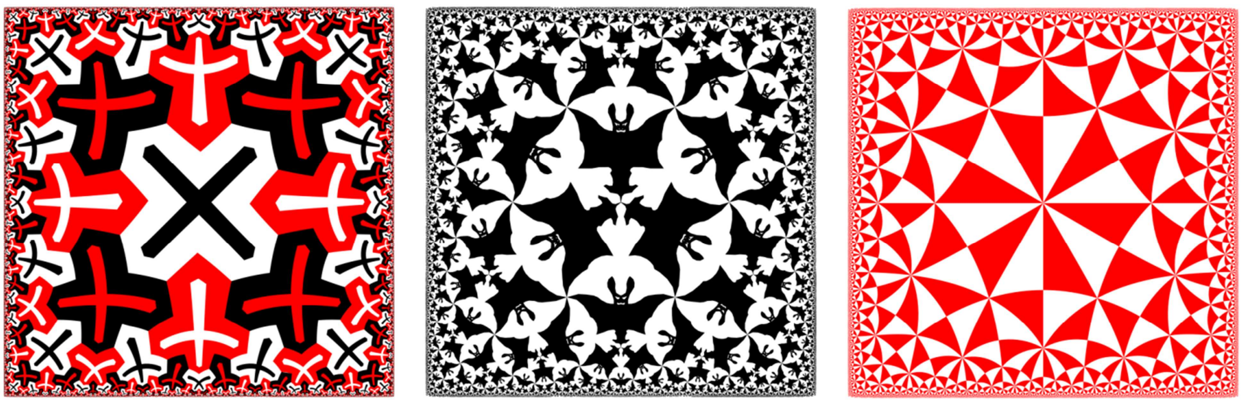

Figure 1: *Square renditions of the Escher's Circle Limit II (left), Circle Limit IV (center), and Coxeter's subdivided {6,4} tessellation (right) of the Poincaré disk*

Introduction

The Poincaré disk model of the hyperbolic plane is probably the most popular model of hyperbolic geometry. In fact, it served as the basis for four of M.C. Escher's "Circle Limit" masterpieces. The Poincaré disk is a conformal model, which means the hyperbolic measure of angle in it is the same as its Euclidean measure [2]. In other words, it does not distort angles. It also has a curious property that it encases an infinite region within the confines of a bounded unit disc. This property is what originally inspired Escher to make his "Circle Limit" woodcuts in the late 1950's. A few decades later, Douglas Dunham pioneered the use of computers for creating hyperbolic art based on the Poincaré disk [2].

Most of the world's paintings are rectangular. People are much more accustomed to seeing rectangular artwork than circular artwork. This is the main motivation of this paper. For artistic reasons, it is useful to have a model of the hyperbolic plane as a square. Also, when display space comes at a premium, such as in rectangular computer screens, it is preferable to avoid circular shapes which do not tile easily and are suboptimal in utilizing display area.

As a matter of fact, Escher himself was interested in encasing an infinite region of patterns within the confines of a square [9]. This is evidenced by his "Square Limit" woodcut shown in Figure 9. We share the same sentiment as Escher. We believe that the myriad of circular artwork based on the Poincaré disk can be reinterpreted as a square. In this paper, we shall discuss three different ways of doing this.

Disc-to-Square Mappings

In 2014, Fong presented and analyzed different sets of equations [4] for mapping the disc to the square and vice versa. In this paper, we shall apply these mappings on the Poincaré disk to produce hyperbolic square regions. The three mappings used in this paper are shown in Figure 3 along with forward and inverse mapping equations. Note that each mapping converts the perimeter of the circle into the perimeter of the square. This rim-to-rim behavior is important in creating viable hyperbolic square regions.

In order to illustrate the visual properties of the mappings, we also included diagrams for a disc with a radial grid converted to a square. This appears on the left side of Figure 3. Similarly, on the right side, we included diagrams for a square grid converted to a circular disc.

It goes without saying that there are infinitely many ways to map a circular disc to a square. For example, there are algorithmic methods that iteratively optimize on some criteria [10]. However, for this paper, we only focus on three invertible mappings that have explicit analytical equations.

Canonical Mapping Space. The canonical space for the mappings presented here is the unit disc centered at the origin with a square circumscribing it. This is shown in Figure 2. This unit disc is defined as the set $\mathcal{D} = \{(u,v) |\, u^2 + v^2 \leq 1\}$. The square is defined as the set $\mathcal{S} = [-1,1]$ x $[-1,1]$. This square has a side of length 2. We shall denote (u,v) as a point in the interior of the unit disc and (x,y) as the corresponding point in the interior of the square after the mapping. The diagram in Figure 3 has equations for mapping (u,v) to (x,y) and vice versa.

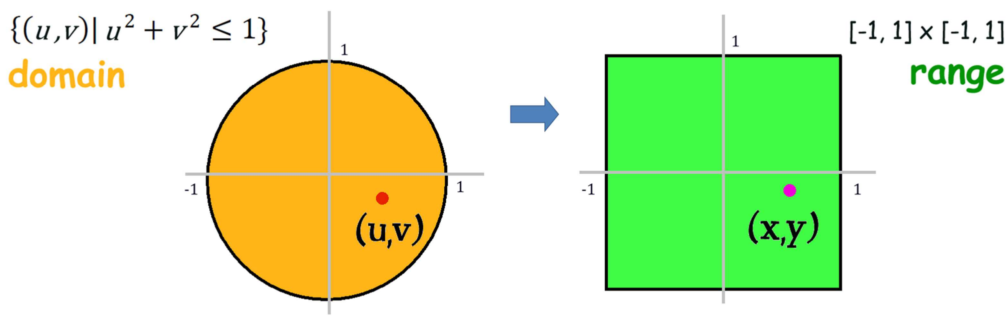

Figure 2: *Canonical mapping space*

Note that for the sake of brevity, we have not singled out cases when there are divisions by zero in the mapping equations. For these special cases, just equate x=u, y=v and vice versa when there is an unwanted division by zero in the equations. This usually happens when u=0 or v=0 or both.

It is important to mention here that the most relevant and appropriate mapping for the converting the Poincaré disk to a square is the *Schwarz-Christoffel mapping*. This is because the mapping is conformal and, hence, a natural extension of the Poincaré disk to the square. In other words, undistorted Euclidean angles in the Poincaré disk remain intact on the square after using this mapping. In fact, all the patterns in Figure 1 are generated using the Schwarz-Christoffel mapping. We shall denote this square mapping of the Poincaré disk as the *conformal hyperbolic square*. The two other mappings covered in this paper are the *FG-squircular* and the *elliptical grid mappings*. We will discuss them all in more detail later.

Hyperbolic Geometry and the Poincaré Disk

There is a long and storied history of non-Euclidean geometry that we will only gloss over here. Non-Euclidean geometry arises from the negation of Euclid's fifth postulate -- also known as the parallel postulate. A modern formulation of this postulate states that "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point". This formulation is known as *Playfair's axiom*. Hyperbolic geometry arises when the axiom is negated with the following statement "In a plane, given a line and a point not on it, there are several lines parallel to the given line than can be drawn through the point".

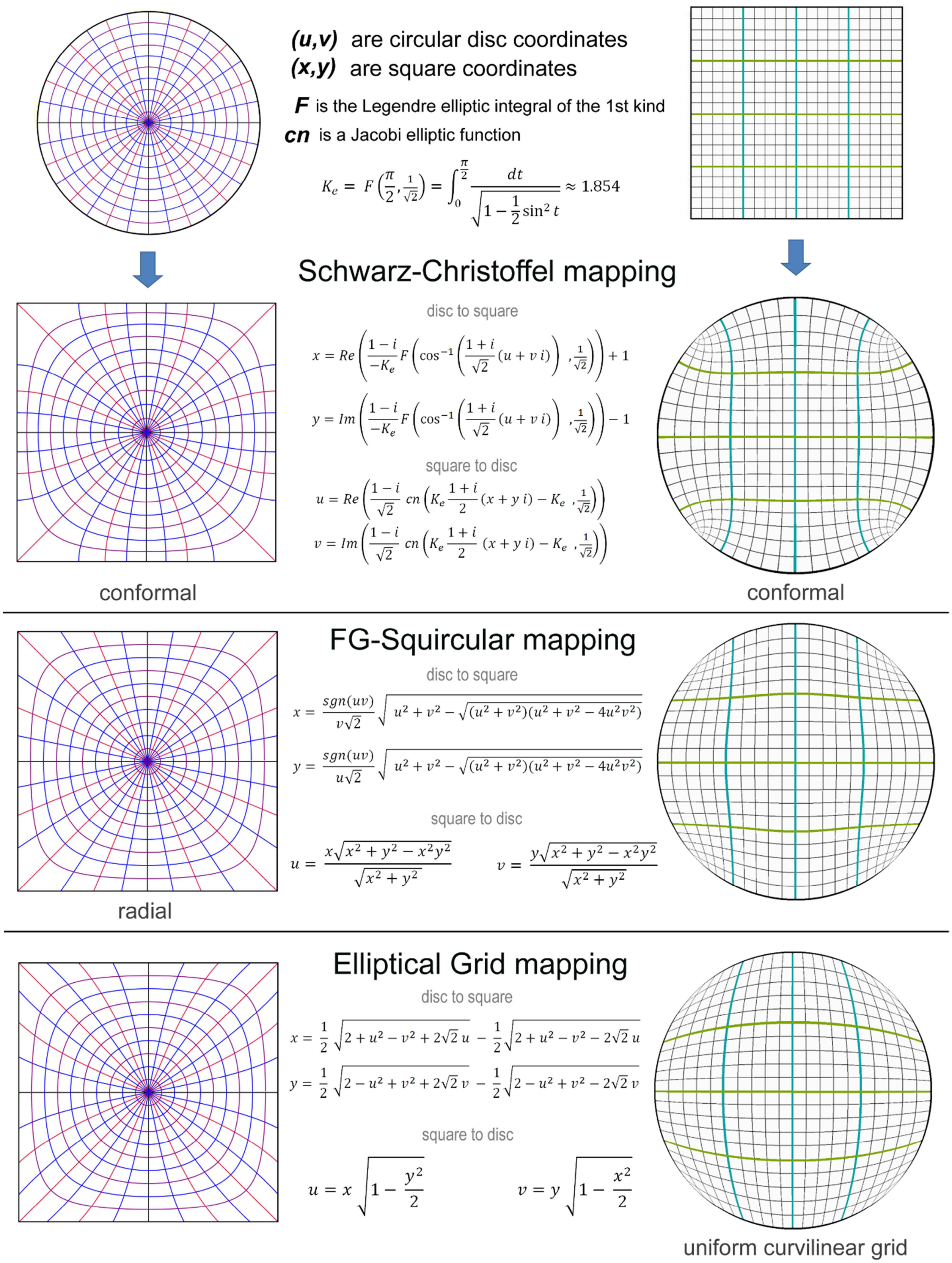

Figure 3: *Some mappings to convert a disc to a square and vice versa*

Models of the Hyperbolic Plane. Mathematicians have come up with many different models of the hyperbolic plane to study hyperbolic geometry. The most well-known and useful models are the Poincaré disk, the Poincaré half-plane, the Beltrami-Klein disk, and the Weierstrass-Minkowski hyperboloid model. These different models are usually taught in undergraduate classes of non-Euclidean geometry. In this paper, we are mainly interested in the Poincaré disk because it is a conformal model embedded as a finite disk on the Euclidean plane. We would, however, like to mention that there are many other exotic models of the hyperbolic plane. These include the Gans model, the hemisphere model, and the Bulatov band model [1]. Moreover, we intend to introduce three more exotic models of the hyperbolic plane by mapping the Poincaré disk to a square.

Using the different mappings shown in Figure 3, we can convert the Poincaré disk into hyperbolic squares. Figure 4 shows a negation of Playfair's axiom on the Poincaré disk and corresponding square mappings. The vertical line at the center stands for a given line and the small red dot stands for a point not on the given line. The colored curves are hyperbolic lines parallel to the given line. We also included a cyan-colored hyperbolic circle on the left side to illustrate the distortion effects of the mappings on circles. Note that the hyperbolic circle on a Poincaré disk is represented by a Euclidean circle.

We make no attempts at proving that these square mappings are equiconsistent models of the hyperbolic plane. Instead, we would like to mention that these mappings are continuous and invertible. Hence they map every point in the hyperbolic plane to a unique point in the square and vice versa. In other words, we believe that the bijective nature of the mappings provides a justification for the models.

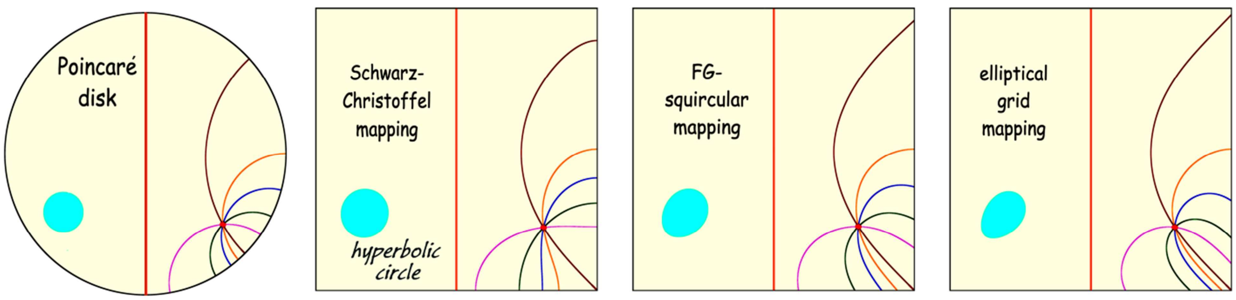

Figure 4: *Playfair's axiom on the Poincaré disk and hyperbolic square regions*

Regular Tilings of the Hyperbolic Plane. Unlike the Euclidean plane, there are an infinite number of ways to tile the hyperbolic plane using regular polygons. In fact, if (p-2)(q-2) > 4, it is possible to get a regular tessellation of the hyperbolic plane consisting of p-sided polygons where q of which meet at each vertex [2]. This type of tessellation is known as a {p,q} tiling in the Schläfli symbol notation. Figure 5 shows some examples of regular tiling on the conformal hyperbolic square

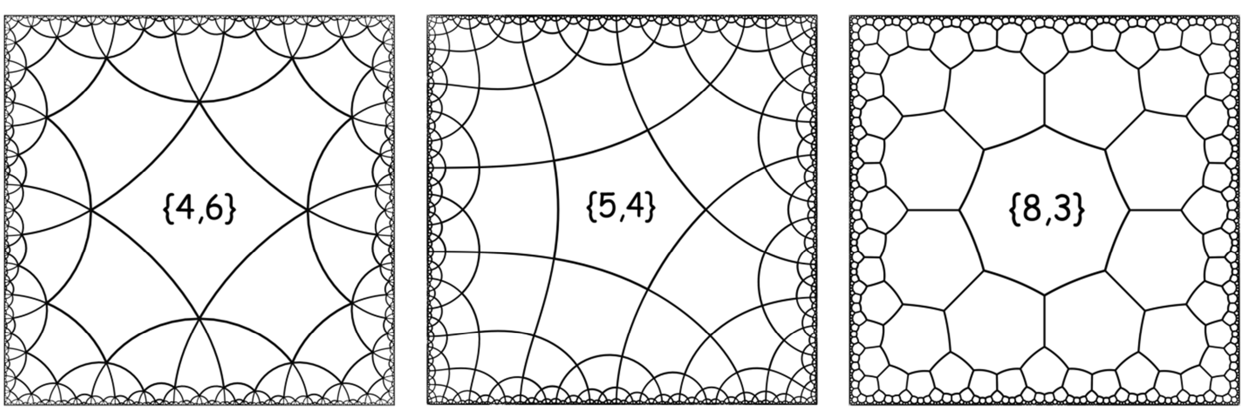

Figure 5: *Some regular tilings of the conformal hyperbolic square*

More Details on the Mappings

Schwarz-Christoffel Mapping. One of the most celebrated results in 19th century complex analysis is the Riemann mapping theorem. It states that for every simply connected subset of the complex plane, there exists a conformal map between this region and the open unit disk. Moreover, it states that this conformal map is unique if we fix a point and the orientation of the mapping.

In theory, the Riemann mapping theorem is nice, but it is only an existence theorem. It does not specify how to find the conformal mapping. The next important breakthrough came with the works of Hermann Schwarz and Elwin Christoffel. In the 1860s, Schwarz and Christoffel independently developed a formula for a conformal mapping between the unit disc and a simple polygonal region in the complex plane. The formula is complicated and involves an integral in the complex plane. Furthermore, for most polygons, the integral can only be approximated numerically. Fortunately, for the special case of the square, the Schwarz-Christoffel formula can be reduced to an explicit analytical expression involving elliptic integrals and elliptic functions [5]. Meanwhile, these special functions can be computed easily using some well-established fast and robust algorithms.

A Fundamental Conformal Map. Without getting much into the mathematical underpinnings of the Schwarz-Christoffel mapping, we show in the figure below a fundamental conformal map between the circular disc and the square in the complex plane. This mapping can be derived by simplifying the Schwarz-Christoffel integral for the square and using the doubly-periodic nature of the Jacobi elliptic function *cn* on the complex plane. In essence, one could map every point inside the unit disc to a square region conformally by just an evaluation of the complex-valued Jacobi elliptic function $cn(z,\frac{1}{\sqrt{2}})$. Furthermore, the inverse of the mapping can be calculated using the Legendre elliptic integral *F*.

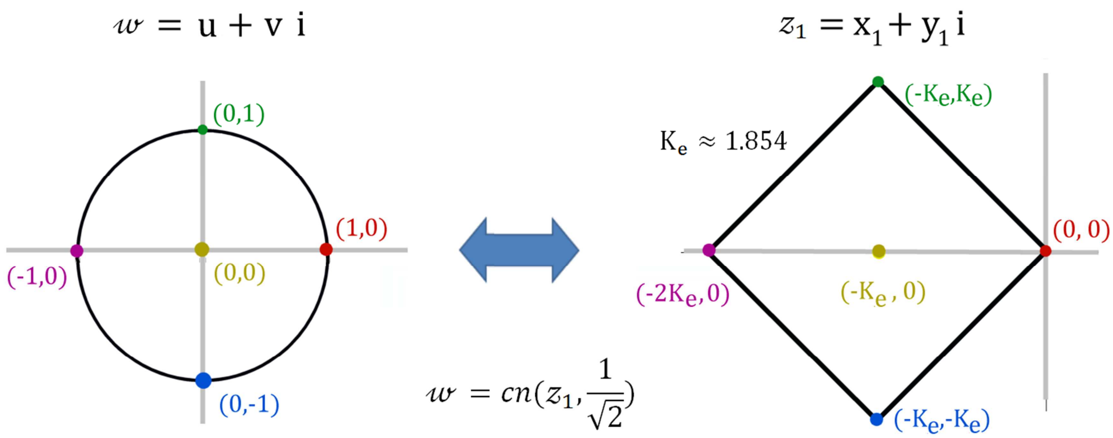

Figure 6: *A conformal map between the disc and square in the complex plane*

Canonical Alignment. The main drawback of the diagram on the complex plane is that x and y coordinates are not in our canonical mapping space. Figure 6 shows a square with corner coordinates in terms of a constant K_e instead of the ±1 that we desire. Moreover, the square is tilted by 45° and off-center from the origin. In order to get the mapping into our canonical mapping space, we need to do a series of affine transformations on the square. This includes centering the square to the origin and scaling it down to have a side length value of 2. In order to do this, we introduce a rotational factor of $\frac{1\pm i}{\sqrt{2}}$ for the 45° tilt as well as K_e offsets and scale factors. This is exactly what happens in the explicit equations that are provided in Figure 3. Basically, we have this canonized mapping equation in the complex plane

$$\boldsymbol{w} = \frac{1-i}{\sqrt{2}}\, cn\left(K_e \frac{1+i}{2}\boldsymbol{z} - K_e, \frac{1}{\sqrt{2}}\right) \qquad where \quad \boldsymbol{w} = u + v\, i \quad and \quad \boldsymbol{z} = x + y\, i$$

and its inverse

$$\boldsymbol{z} = \frac{1-i}{-K_e}\, F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}\boldsymbol{w}\right), \frac{1}{\sqrt{2}}\right) + 1 - i$$

Several people have previously used the Schwarz-Christoffel mapping in conjunction with the Poincaré disk [6][8]. However, we believe that we are the first to provide explicit formulas for converting the Poincaré disk to hyperbolic squares. Moreover, since we focused our attention solely on the square, we are also able to provide inverse equations for the mapping.

Fernandez-Guasti Squircle. In 1992, Manuel Fernandez-Guasti [3] introduced an algebraic equation for representing an intermediate shape between the circle and the square. His equation included a parameter *s* that specifies the squareness of the shape. Figure 7 illustrates the shape at varying values of *s*.

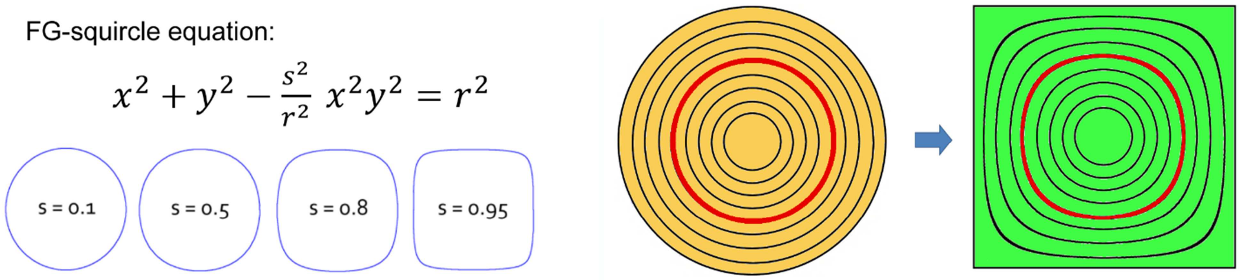

Figure 7: *Fernandez-Guasti squircle (left) and its use in mapping the disc to a square (right)*

The parameter s can have any value between 0 and 1. When $s = 0$, the equation produces a circle with radius r. When $s = 1$, the equation produces a square with a side length of $2r$. In between, the equation produces a smooth curve that resembles both shapes. The two other disc-to-square mappings covered in this paper are based on the Fernandez-Guasti squircle.

FG-Squircular Mapping. In 2014, Fong used the Fernandez-Guasti squircle to come up with a mapping between the circular disc and the square [4]. He designed the mapping with two key constraints. The first constraint is that circular contours inside the interior of the disc be mapped to squircular contours inside the square. This is illustrated in right diagram of Figure 7. The second constraint for the mapping is a radial constraint. This means that points inside the disc will only move radially from the center during the mapping. This is evident by observing the radial grid mapped to a square in Figure 3.

Elliptical Grid Mapping. In 2005, Philip Nowell introduced a square to disc mapping that converts horizontal and vertical lines in the square to elliptical arcs inside a circular region. In effect, this mapping turns a regular rectangular grid into a regular curvilinear grid consisting of elliptical arcs. Nowell provided a mathematical derivation of his mapping in his blog [7]. In 2014, Fong analyzed the mapping and came up with an inverse equation [4]. This effectively made the mapping a bijection between the disc and the square. Fong also showed that the mapping converts circular contours inside the disc to Fernandez-Guasti squircles inside the square [4].

Distortion Comparisons. Figure 8 shows a side-by-side comparison of the three disc-to-square mappings applied to Escher's "Circle Limit I". The two non-conformal mappings are not as pretty as Schwarz-Christoffel mapping when applied to hyperbolic art. This is most evident near the four corners where they appear muddled and fuzzy. Qualitatively, these two mappings produce patterns that look very similar, but they are not the same. One is a radial map and the other is not. Nonetheless, the differences between them are quite subtle and may require a bit of squinting in order to see.

Even though the Schwarz-Christoffel mapping does not distort angles, it does distort area. This is quite evident near the four corners of the square where there is significant size distortion when compared to its circular image. The square grid mapping in Figure 3 illustrates this distortion quite well. Meanwhile, although the two other mappings are not conformal, they make up for it by having less size distortion in the four corners. As a matter of fact, sometimes having less size distortion is more desirable than having less angular distortion. For example, in Figure 10, we show circular mappings of the Monopoly™ board game. The two other non-conformal mappings actually have much more legible corners than the Schwarz-Christoffel mapping.

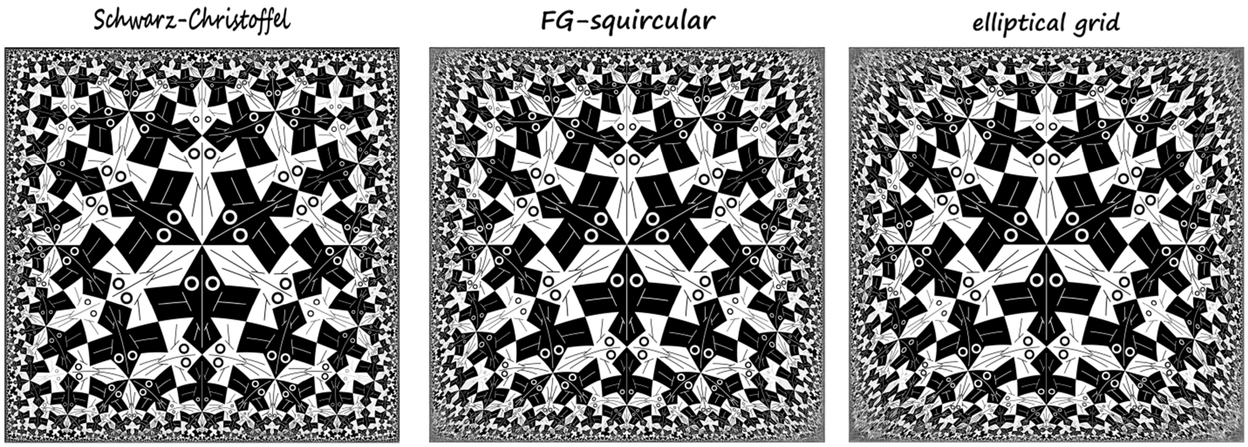

Figure 8: *Different square mappings of Escher's Circle Limit I*

Software Implementation. All of the hyperbolic square figures shown in this paper were generated using software written in C++. The software implementation involves generating hyperbolic points, lines, and polygons inside the Poincaré disk. For this, we used algorithms devised by Dunham [2]. After calculation of vector art coordinates inside the Poincaré disk, we used the disc-to-square mappings provided in Figure 3 to get coordinates inside the square for rendering. Meanwhile, Figures 9 and 10 were generated using an image processing computer program that we wrote in C++. The program reads and writes bitmapped image files and applies the mapping equations to individual pixels. The built-in complex number class library in C++ was very useful in our software implementation. We also found the GNU Scientific Library (GSL) useful for the numerical computation of special functions such as the incomplete Legendre elliptic integral of the 1st kind and Jacobi elliptic functions.

Other Artistic Uses

Although we put emphasis on using the disc-to-square mappings to convert the Poincaré disk to hyperbolic squares, we would like to mention that this is not the only use for these mappings. One can certainly use these mappings to convert other artwork from a disc to square and vice versa. For example, since the mappings are all invertible, we can convert square diagrams into circular ones. This is illustrated in Figure 9 where Escher's "Square Limit" woodcut is converted into a circular disc via the Schwarz-Christoffel mapping. We also show other examples in Figure 10 which includes a rendition of Hilbert's space-filling curve inside a circular disk. We have many more example results available in our http://squircular.blogspot.com website.

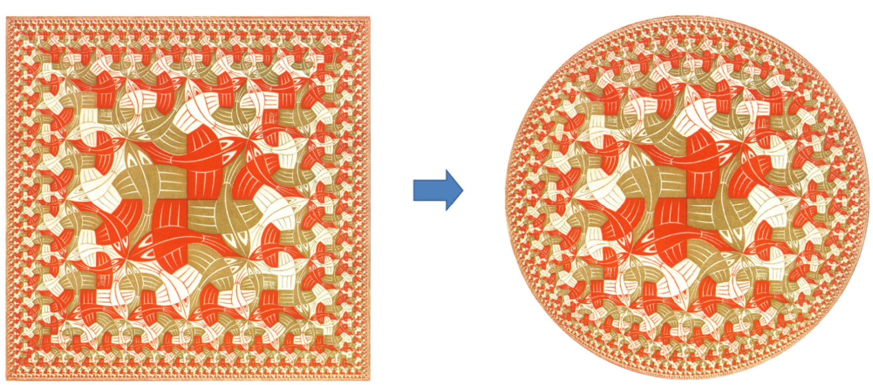

Figure 9: *Escher's Square Limit (1964) converted to a circular disc*

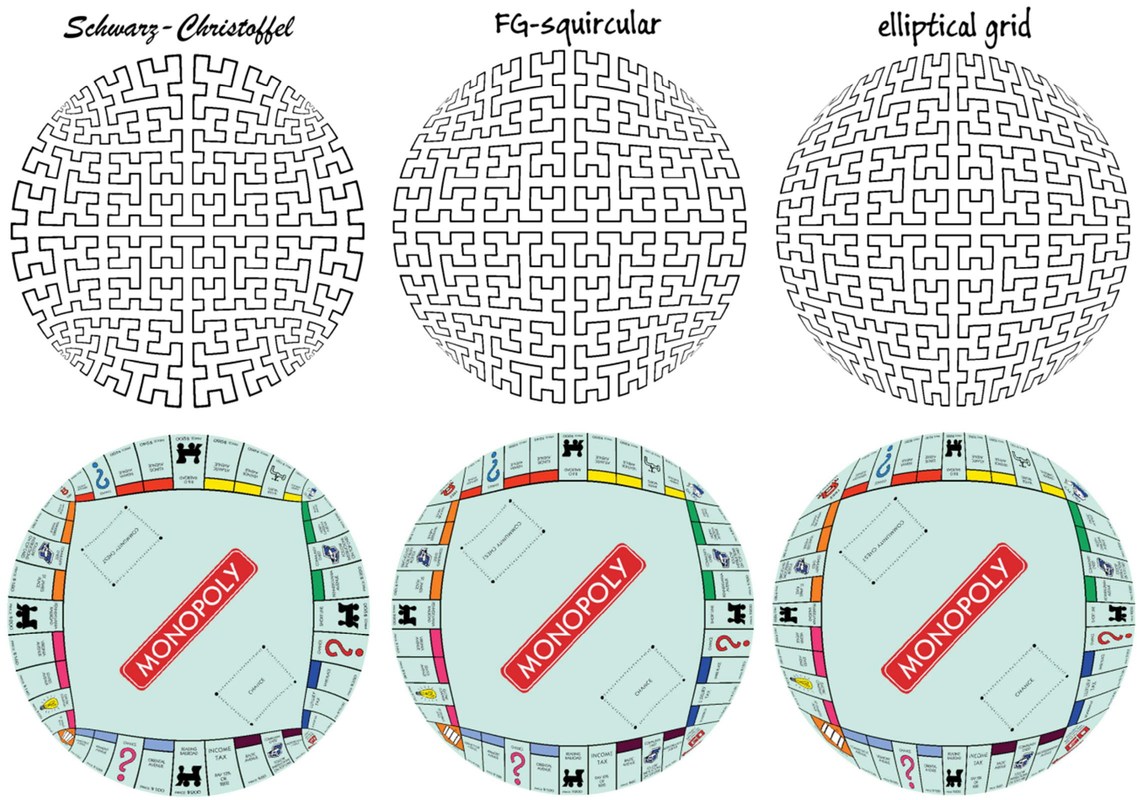

Figure 10: *Circular renditions of Hilbert's space-filling curve (top) and the Monopoly™ board game by Hasbro® (bottom)*

Summary and Conclusion

We presented three explicit mappings for converting circular artwork to squares and vice versa. For hyperbolic art, the Schwarz-Christoffel mapping produces the best results because of its conformal nature. This is not necessarily the case for other types of artwork because the Schwarz-Christoffel mapping has considerable size distortions near the four corners.

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