Artefacts to Enhance Geometrical Thinking

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Abstract

This paper briefly reports on research exploring how pre-service teachers may improve their geometrical thinking while pursuing a realistic mathematics education activity. The participants were encouraged to photograph artefacts that include basic geometrical objects as the first step of the activity. They then created dynamic GeoGebra worksheets inspired by those pictures. Their work and online communication during the activity were explored to understand to what extent they could improve their geometrical thinking. Data suggest that the activity led them think more visually, dynamically and adventurously, and that the project has the potential to foster their creativity.

Introduction

In Turkey we live in an environment full of cultural artefacts, and these artefacts are decorated with lots of geometrical figures, some demonstrating excellent examples of transformational geometry. However many people including students and teachers still struggle with geometry problems, complaining that "I cannot see!" Taking this complaint into consideration I decided to encourage my undergraduates to explore geometrical patterns they could pick from their own environments, and to transform the patterns onto a digital platform, GeoGebra. There are many publications that explore and explain the relationship between mathematics and art, however integrating this relationship in the educational context is relatively limited [1, 2, 4, 5, 7]. Moreover this research is unique because the participants were encouraged to act creatively as explained below.

The theoretical framework of the study stems from the Realistic Mathematics Education (RME). The RME framework suggests that learners may learn better when they take real life problems, move into the world of mathematics through horizontal mathematization, and keep working on the problem in the mathematical world to deepen their knowledge through vertical mathematization [3, 8, 9]. Readers may easily identify two major components of RME in the study: the participants employed horizontal mathematization to identify and transform patterns into a digital environment, and deepened their geometric thinking to elaborate their re-created artefacts into some new forms (vertical mathematization). Although the study was conducted with a number of students, I will describe here only one student and his work in detail, illustrating a clear picture of improvement.

The Study

This is research with data collected in a method course taken by more than 150 pre-service teachers. All the names used here are pseudonyms. The study is a qualitative study exploring *to what extent participants improve their geometrical thinking while interacting with cultural artefacts and the geometry* *located on those artefacts.* Data include pictures and dynamic GeoGebra applets created by participants, as well as their online communication records and interview notes.

In order to understand how they improve their thinking, the differences and similarities between their pictures and dynamic GeoGebra applets were compared. Although their work provided me with some insight into how they might have been able to deliver these experiences to their future students, I needed more information, and went further. I interviewed some selected participants to look at how much they might have understood the idea behind this activity. The following sections present some selected findings drawn from the data.

In order to start working on their own activity the participants were asked to photograph any artefact they could find nearby that contains basic geometric figures or figures looking like basic geometric figures. What I mean by *basic geometric figures* are geometric figures such as squares, polygons, regular stars, circles, and lines that they can expect to introduce to their future students. Since the participants were undergraduate students at a state university, located in a small city in the northeast of Turkey, they were lucky enough to find a number of artefacts: rugs, carpets, head scarfs, earthenware and tiles decorating the outside of buildings as well as inside the masjids.

Once they took their pictures, they were supposed to share them with the course instructor so he could confirm that they contain enough figures, or at least that there is a picture that is good enough to inspire the participant. Since the approval process was done through online collaboration (using pbworks.com) their classmates were also able to comment on each others picture, as well as their GeoGebra work at a later stage. In fact they were encouraged to do so by establishing small working groups. pbworks.com is a wiki-based online platform and free to use for educational purposes. Their communication through pbworks.com also provided me with some information about how their thinking evolved in time. Their way of using the platform included, but was not limited to, collaborative explorations of and reflections on the pictures in terms of the functions of transformational geometry and the basic geometric figures affected by those functions.

I present Kemal's work and his communication with his friends as well as some excerpts from the interview I did. I have no doubt that the data collected from only one student is generalizable. The reader can see that the process presented here is not person-specific, and what I describe serves merely as an example. Nonetheless I have tried to be on the safe side and not to overgeneralize the study. What I do in the following sections is to demonstrate how I analyzed his work (a GeoGebra applet) by comparing it with what they communicated on pbworks.com. Finally, I provide an excerpt from the interview that I conducted to confirm my interpretation.

Starting with a Static Picture. The picture seen in Figure 1 is a column at the university entrance, and the decoration of the column contains squares, rectangles, irregular octagons, trapezoids as well as a combination of these basic objects. The color of the column is light grey as illustrated in the figure, however students were allowed to change the color, and even to elaborate the pattern to create new ones.

Creating a Dynamic Pattern. While reflecting on pictures, participants first attempted to create the patterns as they were. That is their very first reactions to each others work were to identify the basic figures and their positioning. The following excerpt is taken from the communication between Kemal and two other participants, who are in the same classroom but working on different artefacts:

> Reyhan: *Well, see there is a column, at the middle, repeating one square and one rectangle*.
> Murat: *I suggest creating the same column first, then creating an octagon on one side and reflecting over the column*.
> Kemal: *I agree with you. However, why don't I start with semi column and use symmetry?*

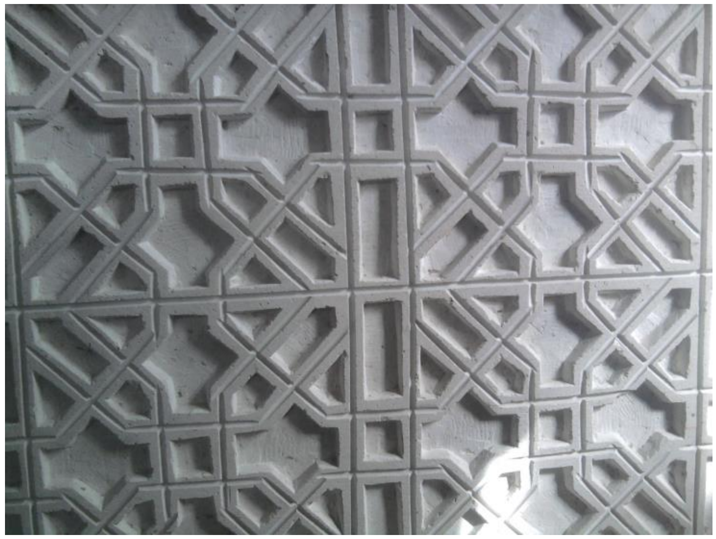

Figure 1: *Kemal's initial picture*

Rather than drawing the picture as it is, participants were encouraged to employ transformational geometry, and to be creative as much as possible. That is they were not supposed to create a dynamic applet very close or similar to the original picture, rather they were encouraged to start with the original picture, prioritize some elements, and create something new. The goal at this stage was to encourage them to explore the picture in as much detail as possible and be inspired by the picture to go further, and *see and illustrate what is unseen on the picture*. The following excerpt illustrates how they employed their creativity:

> Kemal: *Hey, wait a minute! Why don't we simplify the figure?*
> Murat: *How? Having all of them as squares rather than one square and rectangle?*
> Reyhan: *If it is allowed to use different figures, I would go with triangles*!
> Kemal: *That is exactly what I have in mind: Triangles*!

Kemal might then have started to create his applet by employing GeoGebra, one of the well-known Dynamic and Interactive Mathematics Learning Environments (DIMLE) at our university. It seems that he had started his dynamic pattern with three triangles, as they agreed, each having different colors, and then applied transformations on these basic items as seen on the upper left of Figure 2. We do not have information on how they created these basic items. When Kemal was asked at the interview, his answer was, "I do not really remember how I came to this final stage. What I remember is that I had a number of different trials and finally decided to go with these triangles." It seems that Kemal did not develop an abstract design based on geometry but he created some arbitrary triangles intuitively.

Looking at the screenshot illustrated on the upper left of Figure 2 one may assume that participant might have used transformational geometry to create all these pieces. Although there might be various approaches to create them, one possibility here could be to use a reflection over a vertical line as the first transformation, and continue with a translation following a reflection over a horizontal line. Leaving the construction details for another analysis, let us explore how the construction changes while manipulating the slider, which was designed to move from 0° to 360° with an increment of 45°. However Figure 2 illustrates screenshots only when the slider is at 90° (upper right), 180° (bottom left), and 360° (bottom right) because of page limits.

It is rather challenging to imagine what kind of final construct we may have when the slider is at 0° or 90° although it may trigger the spatial imagination of some readers. When the slider is at 180° it becomes clearer that we are going to have new geometrical constructions as we go. For example one can observe the existence of a four-corner star, in white, at the middle of each section, while others may realize that there is a 12-sided polygon at the middle (bottom left). This may serve for teachers as a great point to start talking about geometry rather than the rote learning of a group of rules such as those listed in many geometry books. In fact, we did! When Kemal presented his work to the classroom we had a great opportunity to explore the construction in detail, and how a classroom teacher might use this specific example to introduce basics of geometry to her Grade 1-4 students. The final construction generates a lot more geometric figures to talk about when the slider to gets its maximum value, 360° (bottom right).

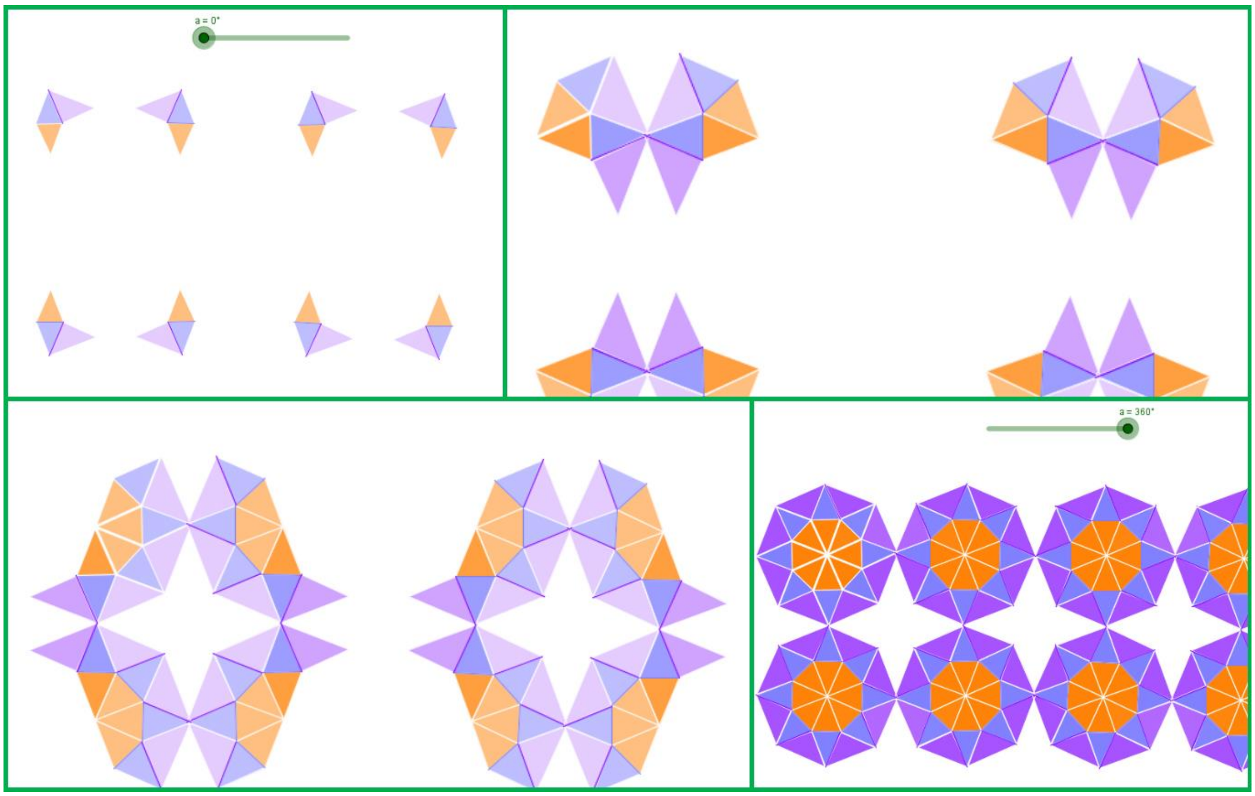

Figure 2: *Screenshots of Kemal's dynamic ornament*

Further Analysis

It was still unclear if the participant deepened his understanding of the activity. In order to reveal what he really had on his mind, an interview was performed. When the participant was asked to reflect on the final construct, he could clearly describe the figures in detail:

> Researcher: *How can you describe what is seen on the screen?*
> Kemal: *There are 8 identical figures, 4 on each row.*
> Researcher: *How do you know that they are identical? Have you measured them all?*
> Kemal: *No, I haven't, but I created one basic figure and applied a couple of reflection functions.*
> Researcher: *What if the reflection function created a slightly different figure?*

Kemal: *Well, impossible! Because the output of reflection should be exactly the same size with the input.*
Researcher: *That is good! What else do you see?*

This excerpt from the dialog between the researcher and the participant clearly suggests that the participant knows the features of transformational functions, at least reflection. His words stating that reflection does not change the size of the figure confirm his understanding of transformational functions as well as his deliberate use of reflection to create patterns.

This is particularly important because they did not get any specific course to learn transformational functions. Rather they were provided very brief information on what these functions are and how they can be created in GeoGebra. They were then expected to improve their own understanding through free exploration. Moreover, their background in geometry was very limited. It was my own observation that their use of formal geometric terminology increased in time. In particular, their understanding of the formal language aligned with their description of figures. The following excerpt shows how this specific participant describes the figures and the relationships among them:

Kemal: *Each basic figure, this figure* [pointing out one of octagons] *is composed of an octagon at the middle and each octagon is made up with eight triangles, identical triangles! Those triangles are also identical because they were obtained by rotating one triangle. Similarly, blue triangles are reflected triangles of the yellow triangles over their bases. These purple triangles* [pointing out purple triangles between blue triangles] *complete the star to another octagon.*
Researcher: *So, you created an octagon based on the first octagon. Would it be possible to tell us the relationship between two octagons? For example, the ratio between their sides or areas!*
Kemal: *Well, I do not know how to calculate. Its area* [the bigger one] *is definitely more than twice of the second, but I do not know how much?*
Researcher: *How do you conclude that it is more than twice?*
Kemal: [giggling]: *Because the area of blue triangles is equal to the yellow ones, and plus purple triangles.*

Pre-service teachers taking this method course are supposed to teach at Grade 1-4 classrooms on their graduation, and therefore developing geometrical thinking is one of the strands they must achieve prior to their professional life so that they will be able to help their future students. In order to help them achieve this goal they were encouraged to go through an experience as briefly explained above. The following section is a reflection on the themes drawn from the study.

Conclusion

It is clear that the specific participant whose work is illustrated in this paper enhanced his geometric thinking without dealing with some prescribed formulas. Rather, he developed some intuitive understanding of the basics of elementary geometry by abstracting the broad symmetry patterns from a few basic shapes. That said, the data suggest that participants can gradually improve their geometrical thinking in such an activity. We have a couple of pieces of evidence to claim this: (1) their online communication evolved over time by demonstrating a deep analysis of figures; (2) final constructs demonstrate a deeper understanding of figures compared to their very first trials and original pictures; (3) the dynamic patterns they had created demonstrate a deep understanding of relationships among the basic figures; and (4) personal communications and interviews with participants and their final exam results demonstrate that many of them had a complete understanding of transformational functions as well as a perfect development of geometrical interpretation of dynamic worksheets, however we could present only a very limited amount of data in this paper.

Prior to this activity participants' knowledge not only of transformational geometry but geometry in general was very limited. This conclusion was drawn from their communication on pbworks.com. Their language describing the geometric figures was informal, their understanding of relationships among geometric figures was almost basic. In terms of van Hiele levels one could say the average was between 1 and 2, however their interaction with figures, their communication while analyzing pictures, and their work on creating GeoGebra applets helped them to develop a deeper and more conceptual understanding of geometry. For example I found it interesting that he had created four pieces, looking like x and y symmetries of the other and asked the student at the interview:

Researcher: *It seems that you have created a four-pieces construction, each of them is x or y symmetry of another. Have you done it on purpose?*

Kemal: *Yeah! Because the original picture is also a four-piece figure. Look at the column* [pointing out the column mentioned above]. *If you take it as y-axis, this will be x-axis* [again, pointing the horizontal line passing through rectangles].

Researcher: *Why do you have 8 pieces then?*

Kemal: *It was just to show that the pattern can go forever. In fact, I wanted to have another two lines below just because of the same reason but did not do it!*

Researcher: *Why not?*

Kemal: *Well, I found this much of design is enough for the activity*[laughing].

Moreover their final dynamic worksheets, as well as their words in the interviews, suggest that they had developed a perfect understanding of technological pedagogical content knowledge (TPACK) framework [6]. The TPACK framework does not suggest only bringing technology, content, and pedagogy together, rather, it suggests an amalgam of these concepts such that one cannot imagine one isolated from others. When we look at participants' artefacts through this perspective we can easily confirm that they use technology to explore and learn geometry.

I think one task missed in this activity could be to ask participants to describe their work step by step as I did above. I believe that working on such a task may improve their understanding of the geometry behind the task. Including more mathematization may help pre-service teachers improve their conceptual understanding, and critically analyze the artefacts they might see around themselves.

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