Underlying Tiles in a 15th Century Mamluk Pattern

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Abstract

An analysis of a 15th century Mamluk marble mosaic pattern reveals an interesting construction method. Almost invisible cut-lines prove there was an underlying pattern upon which the artisan designed the visible pattern. This construction method allowed the artisan to physically strengthen the work and make its marble tiles more stable. We identify the underlying pattern and generate other patterns using the same underlying tiles. We conclude with an exhaustive description of all such patterns.

Introduction

The city of Cairo (from Arabic *al-Qahira*, “the strong”) was the capital of the Mamluk Sultanate, the greatest Islamic empire of the later Middle Ages. Arising from the weakening of the Ayyubid Dynasty in Egypt and Syria in 1250, the Mamluks (Arabic for “owned slaves”), soldier-slaves of Circassian origin, ruled large areas in the Middle East until the Ottoman conquest of Egypt in 1517.

The great wealth of the Mamluks, generated by trade of spices and silk, allowed generous patronage of Mamluk artists, which integrated influences from all parts of the Islamic world at that time, as well as refugees from East and West. Some of the Mamluk architectural principles are still visible today, mainly in Cairo.

Figure 1 shows a rich and lively pattern from the first half of the 15th century that probably adorned the lower register of a wall of an unknown Cairo building, and now presented at the New York Metropolitan Museum of Art. The polychrome marble mosaic is a classic example of the work of the *rassamun* (Arabic for “painters”), designers whose Cairo-based workshops generated and distributed geometric patterns made of various materials for a large variety of purposes. Marble was not common in the areas under Mamluk control, and was acquired mostly through looting. The marble pieces unsuitable for architectural use were used, among other artistic uses, to create inlays such as this [1]. In reality the pattern is rotated by 90°, but for better readability of the paper we present its rotated landscape version.

Constructing the Pattern

We begin our analysis of the pattern by identifying the pattern’s generating polygons, or using the terminology in [2], the pattern’s basic and practical tile. We recall that the basic tile of a pattern is the smallest polygon with which we can regenerate the whole pattern using translations, rotations and reflections (*fundamental domain*); the practical tile is the smallest rectangular tile with that property.

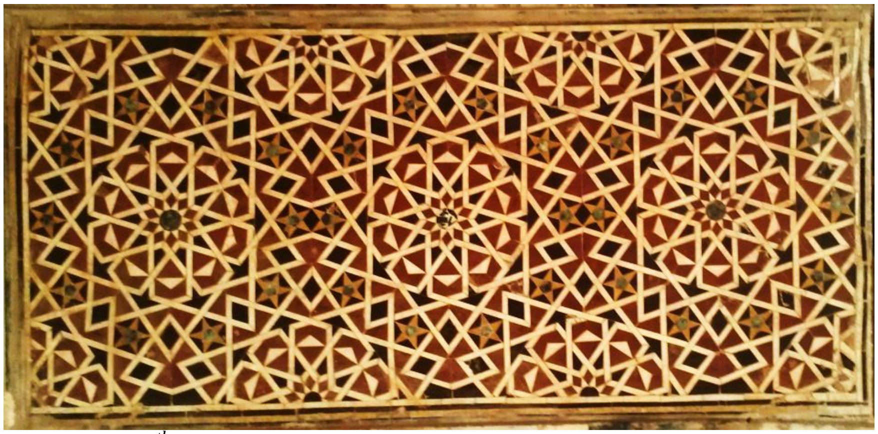

Figure 1: *15th century Mamluk wall panel, made of polychrome marble (rotated). On view in Gallery 454, Metropolitan Museum of Art, New York City, USA. Accession Number 1970.327.8.*

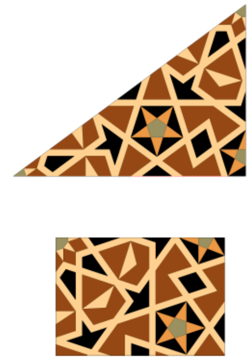

Figure 2: *The pattern's basic tile (top) and practical tile (bottom).*

The pattern exhibits two perpendicular reflection axes, and centers of two-fold rotational symmetry on and off these axes, and so it belongs to the wallpaper group *c2mm* [3]. Hence the basic tile of the pattern is the triangle in Figure 2. Using the techniques described in [2] we find the rectangular practical polygon of the pattern, also in Figure 2. We note that the physical frame of the pattern lies on the reflection axis, so the grid formed by the practical tiles fits perfectly. This means that if the artisan were to create the pattern using the practical tiles, they would fit perfectly in the frame, without needing to break some of them.

A close inspection of the pattern shows that this was not the method the artisan used to lay the pattern. Consider Figure 3, which shows a small fraction of the pattern. Note the tiny cut-lines that run through the marble shapes; they clearly form a near-regular pentagon. As it turns, these pentagons, with the same colorful design drawn upon them, reappear across the pattern, alongside two other shapes – a trapezoid and a regular decagon (incidentally, all three tiles meet at the top left corner of the pentagon in Figure 3). The three underlying tiles are given in Figure 4. Moreover, the frame of the pattern lies on some of the underlying tiles' edges. This gives a good guess about the artisan's method – he created several tiles of each of the three types, and connected them together. This methodology has been previously proved to exist in different times and places in Islamic art, most notably in the use of Girih tiles [4]. It bears mentioning that similarly to Girih tiles, these tiles are also constructible with a straightedge and a compass.

Looking at Figure 3 closely, we see that in fact the pentagon formed by the cut-lines is not closed, but rather, small fractions of edges near its top left and right corners are missing (circled). It seems that the cream colored strapwork strangely disrupted the cut-lines. Looking at the whole pattern again, it can be seen that it was not a one-time mistake, but a rule – every such pentagon was missing parts of its edges near its top left and right corners (without ignoring the tile's rotation, of course). A similar observation can be made about the other two tiles – the decagon and the trapezoid – and in fact, the missing edge on one tile corresponded to a missing edge on its neighboring tile. This shows that the strapwork did not interrupt the underlying tiles at the same places, but rather, it was the artisan who used the strapwork that ran from one tile to the other in order to connect the tiles together. In fact, it can be safely assumed that the artisan created the tiles without the connecting strapwork, and then created separately the strapwork to keep the tiles tight, perhaps even without glue.

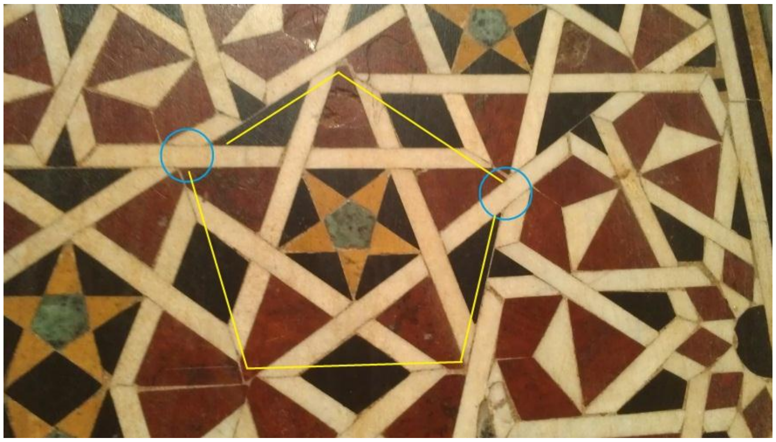

Figure 3: *A close-up on a part of the pattern. Note the nearly invisible cut lines (parallel and close to the yellow lines); they form an almost-regular pentagon. Some parts of the pentagon's edges interrupted by the cream colored strapwork (circled).*

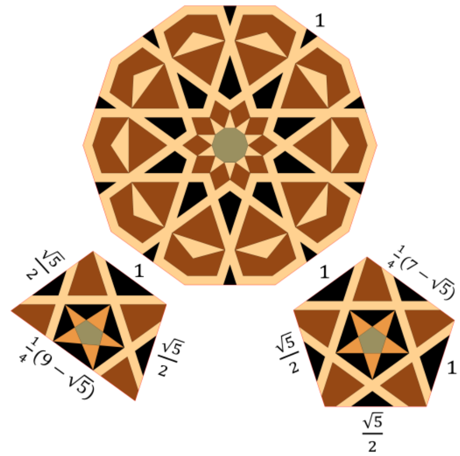

Figure 4: *The pattern's underlying tiles. The pentagonal tile is the one pictured in Figure 3. The edges of length 1 were chosen without loss of generality; the rest follow logically from the tiles' geometry.*

The reconstruction of the pattern, showing just its underlying tiles, is given in Figure 5. For the sake of simplicity and ease of description, from this point on we will ignore the connecting strapwork (red). We will also ignore the design (decoration) of each underlying tile, and may refer to them by their colors in Figure 5 – yellow for the decagon tile, blue for the trapezoidal tile, and green for the near-regular pentagonal tile.

Generating Other Patterns

We shall now look into other possible combinations of the same tiles that yield coherent patterns. Looking at the edges' lengths given in Figure 4, we observe that the blue and the green tiles have both edges of unique length ($\frac{1}{4}(9-\sqrt{5})$ and $\frac{1}{4}(7-\sqrt{5})$, respectively), and we conclude that on the other side of that edge we must have another copy of the same tile, so our tiles de-facto are those in Figure 6[1].

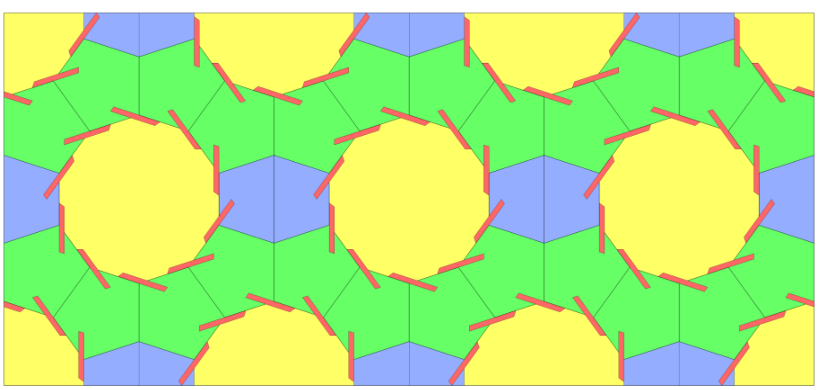

Figure 5: *Reconstruction of the pattern, showing just the underlying pattern and the connecting strapwork (red).*

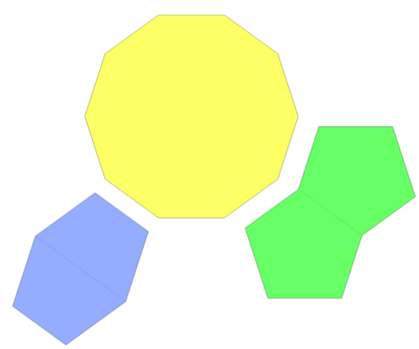

Figure 6: *The underlying tiles with their inevitable neighbors.*

[1] Fortunately, ignoring connecting strapwork, the tiles exhibit reflection symmetry with respect to the bisection of the unique edge, so we can either rotate or reflect the second tile and get the same double-tile.

We note that the green double-tile has two “pockets”, each made of two edges of length 1; since putting any of the green or blue double-tiles’ edges of length 1 next to them results in a 36° gap, which isn’t realizable with these double-tiles, we conclude that we must put a yellow decagon in each of these pockets. A similar argument proves that each of the blue double-tiles’ edges of length 1 must neighbor a yellow decagon, and that there cannot be two adjacent blue tiles around a yellow decagon. In addition, there cannot be two neighboring yellow tiles. Some of these impossible configurations are given in Figure 7.

Our key observation when trying to index all possible patterns with these tiles is counting the number of blue neighbors a yellow decagon has. Since a decagon has ten edges, and each green double-tile uses two of them, we conclude that the number of blue neighbors must be even. Since between any two blue double-tiles we must have at least one green double-tile, we infer that each yellow decagon must have either 0 or 2 blue neighbors.

We shall start with a yellow tile with no blue neighbors. Since no yellow neighbors are allowed either, we must surround the decagon with green tiles (Figure 8.A). Following the first rule we mentioned, we must place yellow tiles in each of the five tiles’ “pockets”; it can be easily seen that only blue tiles fit in between these yellow tiles, and so we get Figure 8.B. We cannot put a blue or a yellow tile next to the freshly placed blue tiles, so we must put green tiles there to get Figure 8.C. Applying the same rules over and over results in Figure 8.D, which can be extended radially ad infinitum, to create a pattern with five-fold rotational symmetry.

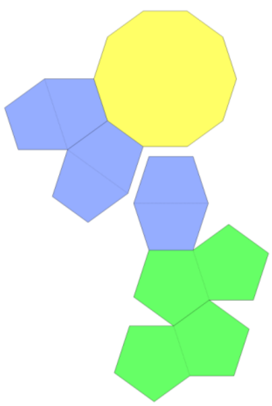

Figure 7: *Impossible tile matching. No tiles fit in the 36° and 72° gaps.*

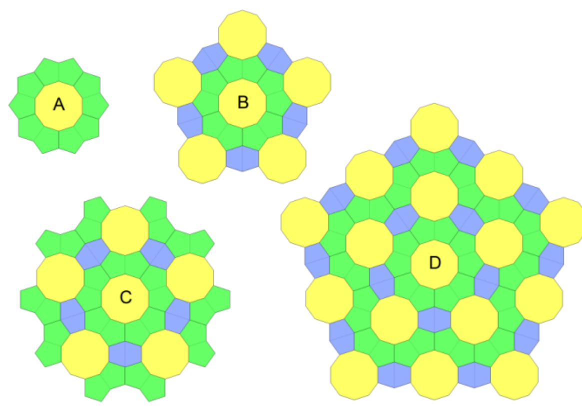

Figure 8: *Construction of a 5-fold pattern.*

Considering a yellow tile with two blue neighbors, we have two options – either they are one green tile apart (we will refer to this combination of type A), or two (type B). In a similar fashion to the previous case, we can extrapolate these initial combinations further, as can be seen in Figure 9. It is noted that while the type B configuration is finite, type A configuration extends infinitely in one direction; however, if we are to avoid a decagon with zero blue neighbors (so as to find patterns different from the five-fold one we found above), we must extend it ad infinitum in both directions.

In order to simplify the discussion, let us now consider the skeleton of such a pattern, i.e. mark the centers of the yellow decagons, and connect two of them with a line if they are neighbors via a blue tile. We shall narrow the discussion to tilings of the plane. Apart from the aforementioned five-fold rotational symmetry pattern, we can use just decagons of type B; this results in the original Mamluk pattern. Otherwise, we are forced to accept at least one decagon of type A, which forces as to accept an infinite row (in both directions) of type A decagons. Using combinations of type A and type B tiles we are able to create any zig-zag pattern with edges lengths as we wish and 108° angles. This gives a pattern repetitive in at least one direction; if there is only one direction, we may discuss the pattern's frieze group [5]. Normally a general pattern's frieze group is *p1*; careful selection of edges lengths and turn points can result in groups *p11m* and *p2*, as we can modify the edge lengths as to get additional symmetries apart from the obvious translational (reflection and glide symmetry in *p11m* and rotational symmetry in *p2*).

Next we consider the more interesting case of patterns with two independent repetition vectors. It first implies that the lengths of the edges in the zig-zag pattern are cycled; let us denote the smallest series upon which the edges lengths are cycled with the vector $\vec{b} = (b_0, \dots b_{n-1})$. For the sake of simplicity we assume that n is even, so that when cycling through the vector we maintain the same directions. An example of our notation is given in Figure 10.

If we impose no condition on the vector $\vec{b}$ we expect the pattern to exhibit only translational symmetry, and so its wallpaper group is the simplest *p1*. However, if we require that $\vec{b}$'s coordinates be symmetrical, i.e., $b_i = b_{n/2-i}$ for all i, then the resulting pattern will exhibit reflection symmetry as well, and so the resulting wallpaper group is *pm*. If we require that all coordinates be constant we also achieve rotational symmetry, and so the wallpaper group is *p2mg*. Alongside the original Mamluk pattern, whose wallpaper group is *c2mm*, these are the only wallpaper groups that may be achieved using these underlying tiles.

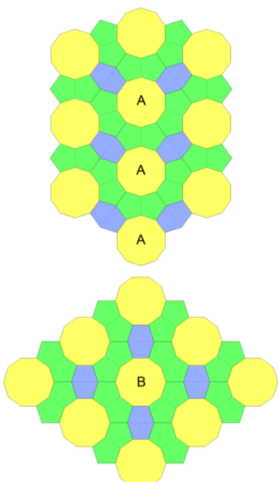

Figure 9: *Decagonal tiles configurations of types A and B.*

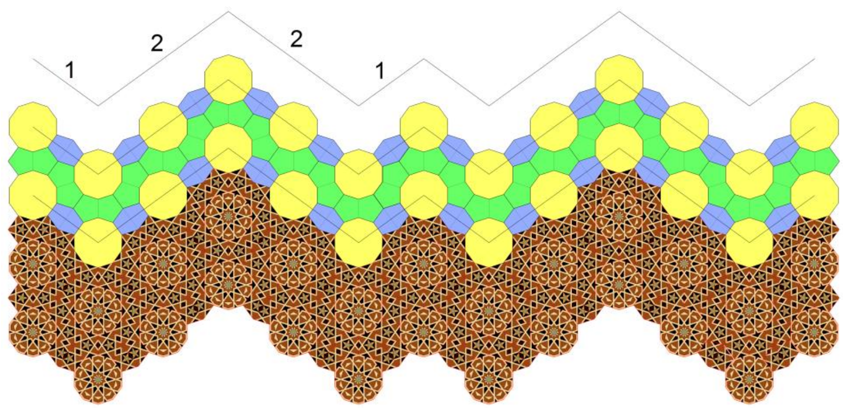

Figure 10: *Skeleton, underlying tiles, and decorated tile of a zig-zag pattern with* $\vec{b} = (1,2,2,1)$.

Summary

By examining nearly invisible cut-lines of a Mamluk marble pattern from the 15th century, we concluded that there is an underlying pattern, whose tiles the artisan decorated and glued together using strapwork as connectors. Despite the different time and place, this approach is similar to the approach seen in several Topkapı Scroll patterns [6], in which one can see the underlying Girih tiles in thin red ink.

Using the underlying tiles, which were the artisan's building blocks, we explored all possible tilings of the plane. We found one pattern with five-fold rotational symmetry, and a rich family of patterns that exhibit translational symmetry in two independent vectors. The possible wallpaper groups of these patterns were identified.

Since the mathematical features of these tiles have been studied exhaustively in this paper, it is suggested that further research should focus on the historical context of the pattern. One question that immediately arises is why the artisan used these underlying tiles; it may be interesting to investigate whether other copies of this pattern, or different patterns using the same underlying tiles, were made near this one. It may also be interesting to look at other patterns from the same time and place and see if they hide an underlying pattern too; and if they do, what is the nature of these underlying patterns.

Another idea that may be worth studying is the decoration of the tiles. It can be seen that because both the blue and green tiles have similar decoration and a strong motif (five pointed star), when looking at the finished work they are almost indistinguishable, and in fact if we look at the decorated pattern of the five-fold pattern we have found (Figure 11, adaptation of Figure 8), the viewer is easily misled into thinking the pattern exhibits translational symmetry, which is obviously forbidden because of the 5-fold rotational symmetry. Such patterns have lately received some academic attention [7], and so this pattern may suggest a novel approach to these types of patterns.

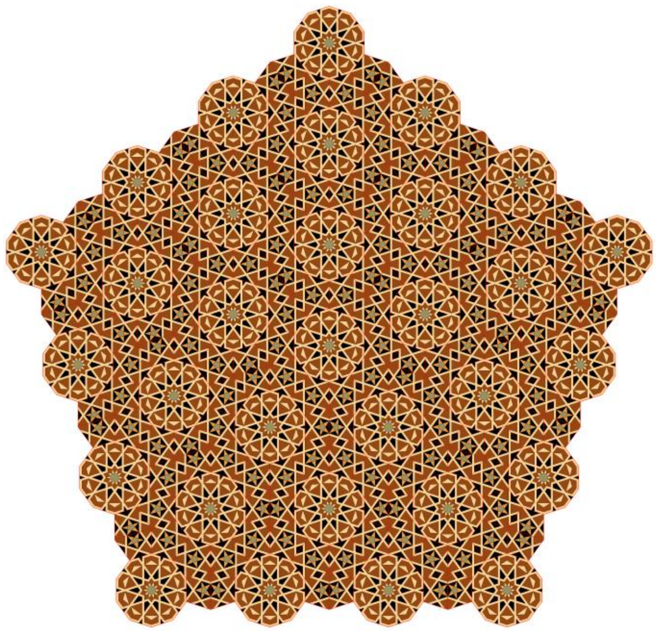

Figure 11: *The decorated five-fold symmetry pattern (adaptation of Figure 8).*

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