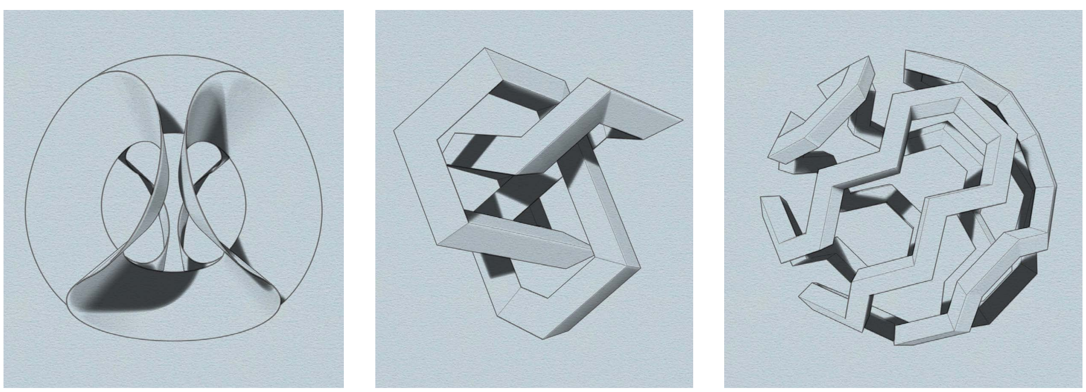
Three Mathematical Sculptures for the Mathematikon

Tom Verhoeff *
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, Netherlands
T.Verhoeff@tue.nl

Koos Verhoeff
Valkenswaard, Netherlands
http://wiskunst.dse.nl

Abstract

Three stainless steel sculptures, designed by Dutch mathematical artist Koos Verhoeff, were installed at the new Mathematikon building of Heidelberg University. *Lobke* consists of six conical segments connected into a single convoluted strip. One side is polished, the other side is matte (blasted), to emphasize the two-sided nature of the strip. The shape derives from an Euler cycle on the octahedron. *Balancing Act* is a figure-eight knot, made from 16 polished triangular beam segments, 4 longer and 12 shorter segments. As a freestanding object it balances on a single short segment. Each beam runs parallel to one of the four main diagonals of a cube. *Hamilton Cycle on Football* is a Hamilton cycle on the traditional football (soccer ball), constructed from 60 matte square beams. Mathematicians know the traditional football as a truncated icosahedron, consisting of 12 pentagons and 20 hexagons, giving rise to 60 vertices.



Lobke *Balancing Act* *Hamilton Cycle on Football*

Figure 1 : *Drawings of the three designs*

1 Introduction

Figure 1 shows renderings of the three mathematical sculptures that were selected for the Mathematikon, the new Mathematics and Computer Science building of Heidelberg University located on the intersection of the Berliner Straße and Im Neuenheimer Feld. Figure 2 shows the original artworks in polyester and wood. In this article, we explain the mathematics in and behind these sculptures.

Koos Verhoeff, who is a mathematician and retired professor in Computer Science at the Erasmus University Rotterdam (Netherlands), designed these sculptures. He started designing and constructing mathematical art in the early 1980s, and has produced hundreds of artworks over the years.

The three sculptures for the Mathematikon were selected in 2013–2014 by Klaus Tschira [4, 10], co-founder of SAP, and the Mathematikon's architect Manfred Bernhardt. The sculptures were constructed in high-grade stainless steel by Geton in Veldhoven, Netherlands [3], and installed at the Mathematikon on Friday 23 October 2015. This project was funded by the Klaus Tschira Foundation [4].

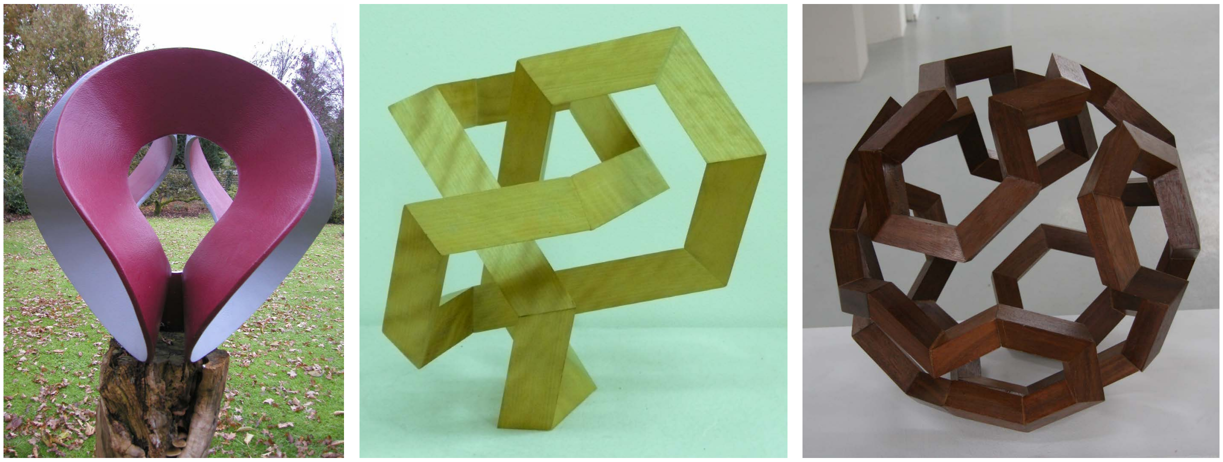


Figure 2 : *Original artworks (71 cm, painted polyester; 32 cm, wood; 50 cm, wood)*

2 Common elements

These sculptures are mathematical in (at least) three ways. First, from their outward *appearance* they look mathematical, even to a layperson. This is betrayed by the regularity of the designs, and the perfectly circular and straight elements. Second, Figure 3 shows some everyday objects that underly these sculptures, viz. a pylon, a figure-eight knot in a piece of rope, and a football (soccer ball). Below each object is shown a corresponding mathematical abstraction, viz. a cone, a figure-eight knot (mathematicians prefer to close the loop, so that it will not ‘accidentally’ untie), and a truncated icosahedron. The second and third sculptures actually were intended to express the mathematical abstraction. Their *message* is mathematlcal. For the first sculpture, the message is less obvious, even to a mathematician, and the cone is merely a means to express it. The third way is that the actual designs require (further) mathematics to make them work; their *construction* is mathematical.

Figure 3 : *Underlying shapes and mathematical abstractions*

In the following sections, we describe each sculpture in more detail. But first, we discuss some more elements they have in common. All three sculptures consist of a single closed loop. In case of *Lobke*, the loop consists of a thin wide strip that curls in space. *Balancing Act* consists of a triangular beam, and *Hamilton Cycle on Football* is constructed from a square beam.

In all three cases, the (two, three, or four) edges of the strip or beam connect up nicely along the loop. This may appear to be an obvious property, but it is not. In general, if one traces out a spatial path by nicely connecting strip cq. beam segments, then at the point of return, the end points of these edges do not meet nicely with the starting points. In particular, the cross section of the strip/beam will rotate as it travels along the path, and need not match up when it returns to the beginning. It requires some math to ensure a proper match. For mathematical details about this torsion, we refer to [6].

The beam segments of *Balancing Act* and of *Hamilton Cycle on Football* are connected by miter joints. In fact, they are regular miter joints [6], where cuts lie in the interior bisector plane of the subtended angle.

3 Lobke

The mathematical construction behind *Lobke* is explained in [8]. In brief, *Lobke* consists of six identical segments of a cone (six *lobes*), where the top angle of the cone and fraction of the cone need to be determined so as to ensure that they can be joined smoothly. The intended mathematical message, however, was actually not revealed there. Since it is less obvious, we explain that message here.

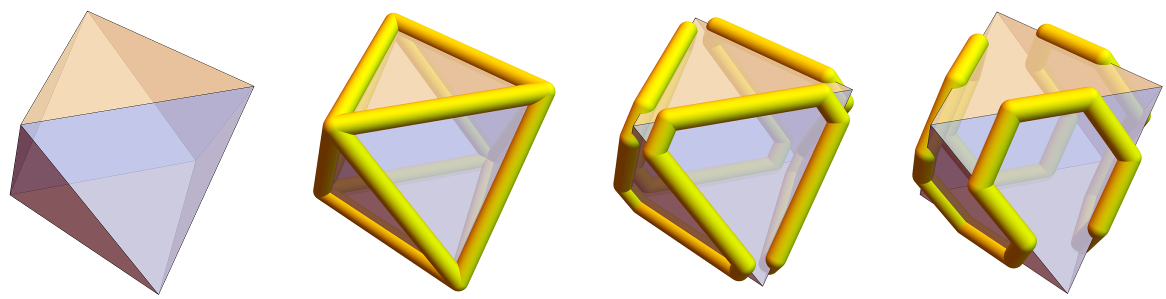


Figure 4 : *Octahedron, octahedron with Euler cycle, cutting corners*

It is well known that a connected graph all of whose vertices have an even degree admits an Euler cycle, that is, a closed path that visits all the edges exactly once. The octahedron (Figure 4, top left) can be viewed as a graph. Each of its six vertices, being incident on four edges, has degree 4, and so it admits an Euler cycle.[1] We have depicted a nice cycle in Figure 4 (second from left). Unfortunately, this way of rendering an Euler cycle does not show in what order it traverses all edges. You cannot see which edges are connected on the cycle, because the cycle visits each vertex twice. In the two right-hand illustrations, these connections are clearly visible, because shortcuts were introduced.

The initial version of *Lobke* arises when the segments are rounded such that they become parts of a circle, and a strip is used as beam. Thus, *Lobke* consists of six conical segments. In fact, the parameters of the cone segments were subsequently tweaked for esthetic reasons to make the top and bottom parts come closer together. The resulting closed strip is two-sided (it has no Möbius twist). To emphasize this, the two sides have a different finish: polished versus blasted (Fig. 5).

Lobke has an order-3 rotational symmetry about the vertical axis, and a 60°-degree rotoreflective symmetry about the vertical axis, interchanging the polished and blasted surfaces. It is also mirror symmetric in three vertical planes. Disregarding the surface finish, its symmetry group is ***223** in orbifold notation [1].

[1] In fact, the octahedron is the only Platonic solid that admits an Euler cycle.

Figure 5: *Lobke at the Mathematikon*

4 Balancing Act

The mathematical message of *Balancing Act* is to portray (the elegance of) the figure-eight knot, as a highly symmetric topological structure. Its mathematical construction is described in [5] by an elegant 3D-turtle-graphics program that generates the figure-eight space walk. The resulting path has constant turn angles and a constant torsion [7] of 60°; it involves two move distances (beam lengths). But there is more to tell.

The figure-eight knot can be embedded in space such that it has four congruent bends that link pairwise (see Figure 6, left). To make the knot as three-dimensional as possible, all angles between the four center strands, connecting the four bends, should be made as equal as possible. In space, the four main diagonals of the cube have this property (see Figure 6, center).

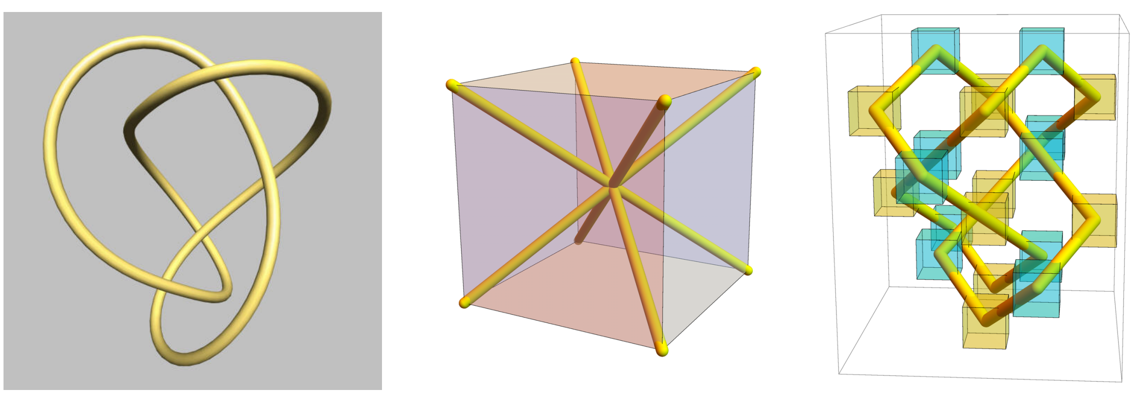


Figure 6: *Symmetrical embedding of figure-eight knot (left); main diagonals of cube (center); figure-eight knot in bcc lattice (right)*

In the body-centered cubic (bcc) lattice [9], which underlies the crystal structure of diamond, the vertices are connected in the directions of the main diagonals of the cube. Figure 6 (right) shows the figure-eight knot embedded in the bcc lattice with 16 straight segments, where all vertices have integer coordinates.

To make the sculpture balance on a short segment, the triangular beam was appropriately rotated along its longitudinal axis (placed on other segments, it will topple). The symmetry group of the path with round beam is **2x** in orbifold notation [1], including a 90°-degree rotoreflection (the figure-eight knot is achiral).

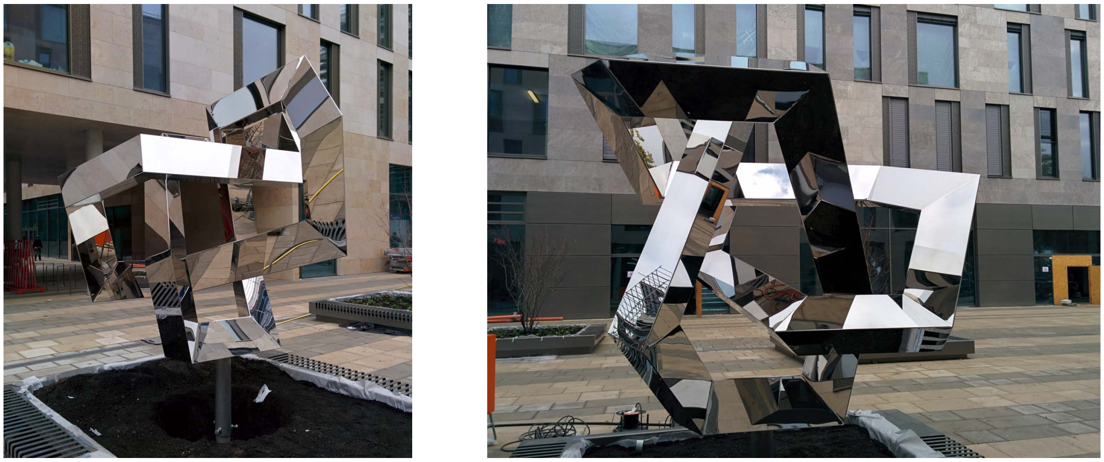


Figure 7 : *Balancing Act at the Mathematikon*

5 Hamilton Cycle on Football

The *Hamilton Cycle on Football* was a personal favorite of Klaus Tschira. Recall that a Hamilton cycle visits all vertices exactly once, possibly (and often necessarily) not visiting some edges. The traditional football is based on a truncated icosahedron consisting of 12 regular pentagons and 20 regular hexagons, thus having 60 vertices and 90 edges. It admits many Hamilton cycles, some more symmetric than others. Each such cycle uses only 60 of the 90 edges, thus leaving 30 edges unvisited.

The Hamilton cycle in this sculpture is the most symmetric, having an order-3 rotational symmetry about the vertical axis (like *Lobke*), and two-fold rotational symmetries about three horizontal axes. Unlike *Lobke*, it is not mirror symmetric. Its symmetry group is **223** in orbifold notation [1].

Figure 8 : *Hamilton Cycle on Football at the Mathematikon, with the artist*

6 Conclusion

In conclusion, we mention some physical facts. *Lobke* is 3.5 m tall and weighs almost 1200 kg. It involved some 300 hours of polishing. *Balancing Act* and *Hamilton Cycle on Football* each have a diameter of 2.4 m and weigh about 350 kg. Their size inspires awe. Figure 9 provides an overview of the three mathematical sculptures at the Mathematikon. We hope that they will inspire the viewers to wonder about mathematics and its beauty, since this is the main motivation of the artist Koos Verhoeff to create such sculptures.

Figure 9 : *The family of three mathematical sculptures at the Mathematikon, together with the artist*

References

[1] John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss. *The Symmetries of Things*. AK Peters, 2008.

[2] Foundation MathArt Koos Verhoeff (Stichting Wiskunst Koos Verhoeff). `wiskunst.dse.nl`

[3] Roestvrijstaalindustrie Geton, Veldhoven, Netherlands. URL: `www.geton.nl/en`

[4] Klaus Tschira (founder). *Klaus Tschira Foundation*. URL: `www.klaus-tschira-stiftung.de`

[5] Tom Verhoeff. "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry". *Int. J. of Arts and Technology*, **3**(2/3):288–319 (2010).

[6] Tom Verhoeff, Koos Verhoeff. "The Mathematics of Mitering and Its Artful Application", *Bridges Leeuwarden: Mathematics, Music, Art, Architecture, Culture*, pp. 225–234, 2008. URL: `archive.bridgesmathart.org/2008/bridges2008-225.html`

[7] Tom Verhoeff, Koos Verhoeff. "Branching Miter Joints: Principles and Artwork". In: George W. Hart, Reza Sarhangi (Eds.), *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture*. Tessellations Publishing, pp.27–34, July 2010.

[8] Tom Verhoeff, Koos Verhoeff. "Lobke, and Other Constructions from Conical Segments", *Bridges Seoul: Mathematics, Music, Art, Architecture, Culture*, pp. 309–316, 2014. URL: `archive.bridgesmathart.org/2014/bridges2014-309.html`

[9] Wikipedia. "Cubic Crystal System", URL: `en.wikipedia.org/wiki/Cubic_crystal_system`

[10] Wikipedia. "Klaus Tschira", URL: `en.wikipedia.org/wiki/Klaus_Tschira`