Permutations of the Octagon: An Aesthetic-Mathematical Dialectic

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Abstract

Artist James Mai employs permutational and combinatorial methods to produce sets of geometric forms for inclusion in his paintings and digital prints. Recent art works include form-sets comprised of form-variants derived from the regular octagon. The artist explains the process by which he creates the form-sets, the geometric features of the form-variants that constitute each form-set, and how the form-sets are composed in art works. In addition to first-order characteristics related to permutation rules, second-order characteristics of symmetry are found in the form-sets and are included in the art works. The permutational and symmetry characteristics of the form-sets are intended as the principal content of the art works and as such are designed to be visually comprehensible, apart from verbal or mathematical explanation. To that end, both mathematical and visual-aesthetic requirements influence the development of permutational form-sets from the start. This "aesthetic-mathematical dialectic" is critical to the development of art works whose mathematical content is adapted and responsive to visual perception.

Introduction and Terminology

During the past 10 or more years, I have become increasingly interested in using permutational methods to produce geometric forms to populate the compositions of my art works. They interest me in part because permutations invite careful visual examination of shapes and discovery of the mathematical order that binds those shapes together—a perceptual challenge for both the artist and the viewer. Permutational forms and their symmetry characteristics are more than merely attributes of a geometric-abstract style in my work; they are the primary content of the art works, intended to be recognized and reflected upon by the viewer. To that end, the forms that I employ in my paintings are determined not only by the mathematical requirements of permutation, but also by the aesthetic requirements of visual perception. That is, the mathematical characteristics are adapted to the capabilities and limits of visual perception. This paper will examine: (1) the permutational development of forms derived from an octagon; (2) the manner by which mathematical order and aesthetic order are brought into balance during the development of the octagon form-sets; and (3) the selections and adaptations of form-sets for composition in some example art works.

For clarity in this paper, I shall employ specific terms for some recurring references to objects and their relationships. The word *form* will be used in a variety of ways: *Form*, without a modifier, will refer to the shape element employed in the finished art works (i.e., paintings are composed of *forms*); these forms are developed from permutational procedures, which will be referred to by compound words that modify the term. *Base-form* refers to the shape that possesses all geometric features found across all permutations (e.g., the base-forms discussed in this paper are equivalent to the complete graphs on the set of vertices of the regular hexagon and the regular octagon). *Form-variant* refers to each shape resulting from the permutation rules; it is always comprised of a selection and arrangement of the features found in the base-form. *Form-set*, or sometimes *minimum form-set*, will refer to the complete collection of form-variants generated by the permutation rules, after the elimination of any form-variants that are the same as another by reflection or rotation—these eliminated forms I refer to as *symmetric redundancies*. *First-order characteristics* are those geometric features of the form-variants that result directly from the permutation rules. *Second-order characteristics* are geometric features of the form-variants that emerge from but are not defined by the permutation rules (e.g., form-variants may be ordered by similarity of symmetry, even though symmetry is not among the permutation rules). *Composition* will usually refer to the specific organization of shapes on the two-dimensional surface (or *picture-plane*, as it is often referred to in the arts) of the art work, which includes relationships of location, scale, alignment, and orientation, not only among forms, but also between forms and the picture-plane. *Aesthetic* will be used in a specific way in this paper, referring to the apprehension of phenomena by the senses. Our interpretation of the word stays close to the Greek origin, *aisthetikos* "of sense perception" and *aisthanesthai* "to perceive." To differentiate our usage from current dictionary definitions related to beauty and what pleases the senses, I will employ the compound word, *visual-aesthetic*. For our purposes, then, *visual-aesthetic* should be understood to mean the specific formal conditions by which the visual sense organizes and apprehends phenomena, especially in determining composition in art works. Since color a vital part of the visual-aesthetic strategies, the reader is encouraged to consult the color version of this paper.

Overview of the Development of Form-Sets

Many of the art works I have produced since 2003 have included geometric form-sets developed by permutational methods. All of these form-sets, including those derived from the octagon that are the topic of this paper, were developed by generally similar methods and stages. It will be useful to briefly describe these methods and stages because they are directly linked to the visual-aesthetic purposes behind the art works. The mathematical characteristics and the visual-aesthetic characteristics are developed in tandem, the needs of each influencing decisions about the other; more will be said about this mutual development in following sections.

A brief summary of my method for making form-sets is as follows: (1) determine a base-form (usually a regular polygon); (2) determine the variable features of the base-form (usually vertices and line segments) and the permutation rules (usually all possibilities of connecting vertices with line segments, which yield different configurations of lines and/or closed shapes); (3) in an orderly set of diagrams, draw every form-variant that would result from the permutation rules; (4) review the resulting array of form-variants and discard all symmetric redundancies (i.e., keep only a single version of any group of form-variants that are alike under rotation or reflection); (5) collect all the unique form-variants into the minimum form-set (the complete set of unique form-variants); (6) examine the minimum form-set for second-order characteristics (e.g., shared features of symmetry or similar component shapes); (7) compose and color the form-variants in an art work such that first- and second-order characteristics may be recognized; (8) consider figurative subject matters when determining composition and colors.

That I employ similar methods and stages to develop the permutational form-sets is not to say that my methods are strictly mathematical. I am trained as an artist and not as a mathematician, so my techniques for generating the form-sets are visual rather than symbolic, and my process is more inductive than deductive. I produce these form-sets by hand-drawing the form-variants with pencil on paper and/or with a digital stylus and vector-graphics software. I do not employ computer algorithms to automatically produce the variants, but rely instead upon a combination of orderly diagramming procedures and visual recognition. At the earliest possible stage, I look for a systematic procedure for varying the features of the base form, such that all form-variants will be found and the drawings will reflect the permutational sequence used to generate the form-variants. This direct involvement in drawing and evaluating each form-variant, although certainly not as fast as using a computer algorithm, has two important benefits for my visual-aesthetic purposes: (1) it lends me an intimate familiarity with each form-variant and its individual characteristics; (2) it usually constrains the number of form-variants, and thus the form-set, to a perceptually manageable number. On this latter point, in other words, the number of permutational variables and the size of the resulting form-set are relatively small, not only because they are being produced "manually" (rather than automatically with a computer) but also because their specific characteristics and relationships are destined to be apprehensible to visual perception. More about this aesthetic purpose follows.

Visual-Aesthetic Constraints on the Development of Form-Sets

From the outset, the development of form-sets is guided by the knowledge that the forms will be used in the compositions of art works. More to the point, the form-sets are developed with the assumption that the form-variants, as well as the permutational basis for their construction, should be understandable to the viewer by close visual analysis of the art work. The visual-aesthetic requirements influence the qualities of the base-form and the number of form-variants, such that the viewer may come to know the mathematical order through the language of vision, apart from verbal or mathematical explanations. It might be thought of as knowledge achieved through the "logic of the eye" instead of the logic of symbolic language, whether verbal or mathematical.

The five most important characteristics related to the form-variants and form-sets that I intend to be discernible to the viewer are: (1) the <u>similarity</u> of shared geometric features among all form-variants (and possibly the ability to infer a common base-form and its permutation rules); (2) the <u>distinctness</u> of each form-variant as a unique arrangement of the geometric features (each form is a permutation of that inferred base-form); (3) the <u>non-redundancy</u> of form-variants (no form is repeated, regardless of rotation or reflection); (4) the <u>completeness</u> of the form-set (the form-variants constitute all possibilities under the permutation rules); (5) <u>second-order characteristics</u> of geometric features among the form-variants (i.e., if the geometric features directly related to the permutation rules are considered first-order characteristics, then second-order characteristics are any geometric features that permit additional, alternative ordering of the form-variants, such as symmetry characteristics). To make these characteristics visually understandable requires their careful adaptation to the requirements of visual art and perception.

As briefly mentioned in the *Introduction and Terminology* section above, *visual-aesthetic* refers to the specific characteristics and organizations of phenomena apprehensible to visual perception. Understood this way, the aesthetic principles of composition in visual art are largely objective, constituting a visual grammar with a complex vocabulary and syntax rooted in the principles of visual perception. For the past approximately 80 years, perceptual psychology and vision science have found considerable experimental evidence to verify and explain many of the visual principles employed by artists for millennia. In vision science, these principles include the classical Gestalt principles of visual organization, which articulate innate perceptual mechanisms for grouping by size, proximity, and orientation, for differentiating figures and grounds, and for recognizing symmetries, edge-alignments, and inferred closure of incomplete shapes [2]. With these and other contributions by perception science, the contemporary artist composes art works aided by knowledge of the structures and functions of visual perception—these are the visual-aesthetic principles that inform and influence my decisions throughout the development of form-sets.

One of the first visual-aesthetic considerations is the number of form-variants that will be produced by permuting a base-form. I have found that, generally speaking, 36 form-variants are near the upper limit for visual recognition of the five permutational characteristics described above (similarity, distinctness, non-redundancy, completeness, and second-order characteristics); more often I prefer to compose with between 8 and 24 form-variants. In the early stages of development, then, I concentrate upon finding the optimal number of variables that will yield minimum form-sets (i.e., after elimination of symmetric redundancies) of no more than about 36 form-variants. When form-sets exceed that target, and this is often the case in my permutation work, I then look for subsets within the form-set that will meet the visual-aesthetic needs of the art work. In addition to the quantitative considerations, there are also qualitative considerations, including visually distinct relationships of size, orientation, alignment, and more, related to Gestalt principles. Perhaps the most important of these for our examination is symmetry. In selecting form-variants for an art work, asymmetrical forms present a greater challenge for visual analysis than symmetrical forms; there are circumstances that can tolerate the visual complexity of asymmetry, but more often those above-mentioned five characteristics are more clearly understood when looking at symmetrical forms. Thus, my visual-aesthetic priorities favor the selection of form-sets (or subsets) with more rather than fewer symmetrical form-variants.

Octagon "Circuit-Form" Permutations and Related Art Work

As background to this examination of the "circuit-form" variants of the octagon, I will briefly explain that in 1999 I produced the same kind of permutations with the regular hexagon, which were realized in a 2000 painting, *Permutations: Astral* [1]. The base-form was a convex hexagon with each vertex connected by line segments to all other vertices. The form-set is comprised of every permutation of six line segments connecting all vertices such that each vertex is the end-point of exactly two lines. Each form-variant can be drawn as a continuous circuit without lifting the pencil from the paper. The set of circuit-forms might be thought of as every continuous linear permutation from the regular convex hexagon to the star hexagram. After the elimination of symmetric redundancies, the minimum form-set was comprised of 12 distinct hexagon circuit-forms. For my purposes, the hexagon form-set struck an ideal balance between the mathematical and the visual-aesthetic requirements. Through careful visual analysis alone, one can discern the five permutational characteristics described in the previous section.

In subsequent years, I developed other kinds of permutational form-sets not based upon polygons, but in 2010 I returned to this subject with the purpose of finding the form-set of linear circuit-forms for the regular octagon. The form-variants of the octagon were constructed with exactly the same permutation rules as the hexagon mentioned above. While I anticipated more permutations resulting from the additional two vertices and two line segments, I was surprised by the dramatic increase in the number of permutations and symmetric redundancies, and the large size of the minimum form-set (Figure 1). Even after the elimination of symmetric redundancies, there are 202 unique form-variants, a form-set that is far too large for my visual-aesthetic purposes. In addition, the individual form-variants are rather complex, especially those that are asymmetrical, and so present more difficulties for visual recognition of both the permutational relationships among the forms and the completeness of the form-set. Nevertheless, the form-set yielded some subsets that proved adaptable to art works; these subsets were comprised of symmetrical form-variants and in small enough numbers to fulfill the visual-aesthetic needs of the art works. The asymmetrical forms, which comprise the majority of the form-set, may prove useful in future, but have so far lain dormant for reasons of visual complexity and similarity of appearance.

The subset of nine form-variants with rotational symmetry was employed for the art work, *Stellar (Octets – Rotational)* in 2011 (Figure 2). The colored circles behind each form indicate the number of extant, outer octagon edges (i.e., boundary edges): two yellow-circle forms possess four outer edges; five red-circle forms possess two outer edges; two blue-circle forms possess no outer edges. These correspond to the row labeled "Rotation" in the table of the form-set (Figure 1). The three color-coded groups of circuit-forms in the art work are arranged in columns in Figure 3. It is notable that such a large form-set as this generates only nine forms with rotational symmetry, and that these are concentrated in only three "outer edge" groups. The nine forms in the art work are composed in a rotationally symmetric array in the 3x3 square group, echoing in the larger composition the rotational character of each individual form.

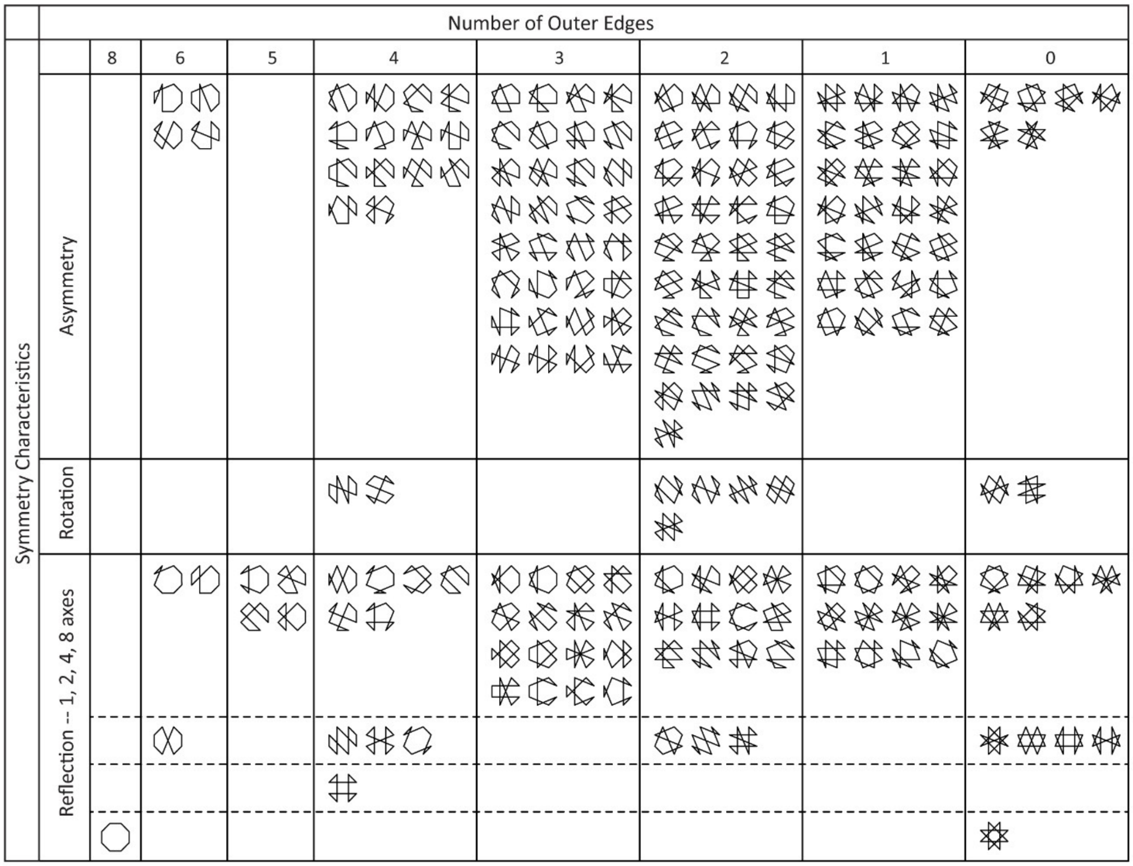

Figure 1: *Form-set of "circuit-form" permutations of the octagon.*

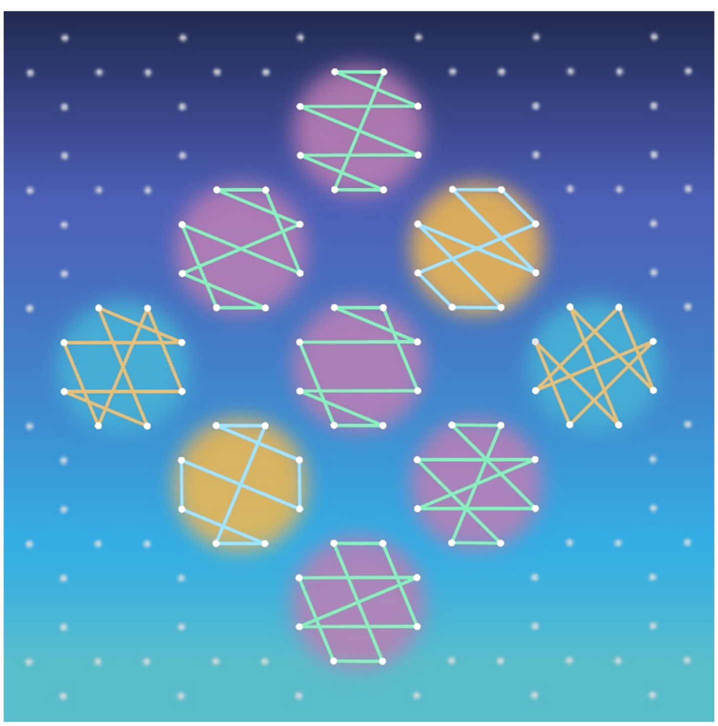

Figure 2: *"Stellar (Octets – Rotational)" digital print, 2011.*

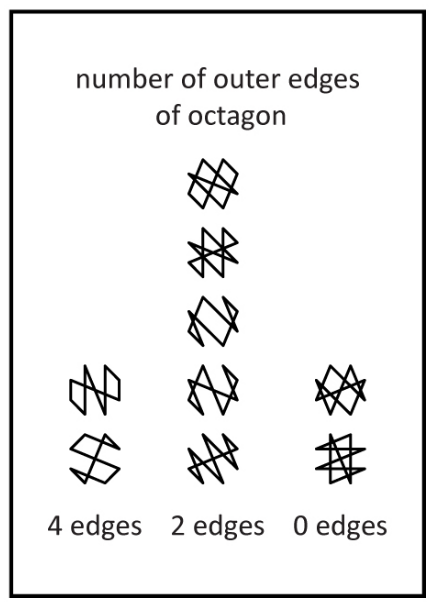

Figure 3: *Form-variant groups indicated by colors in figure 2.*

Octagon "Point-Line-Plane" Permutations and Related Art Works

In an effort to generate a form-set more conducive to visual-aesthetic requirements, I revisited the same octagon base-form with a new set of permutation rules. In this new approach, I abandoned linear circuits and instead emphasized 3 distinct features of the octagon base-form: (1) *points*, which are the eight vertices, treated as autonomous graphic "dots" apart from any function as line-ends or shape vertices; (2) *lines*, which connect pairs of vertices but can no longer join with another line at a common vertex (i.e. the maximum number of lines in the octagon is four); and (3) *planes*, which are closed forms whose vertices are limited to the vertices of the octagon. All permutations must use all eight vertices of the octagon in one of its three roles: as an autonomous point, as a terminus of a line, or as a vertex of a plane. Permutations may include any combination of points, lines, and/or planes.

Yielding 359 form-variants, the resulting minimum form-set (Figure 4) is significantly larger than the circuit-form set (Figure 1); but, paradoxically, this form-set is more adaptable than the circuit form-set for use in art works. This is because the "point-line-plane" form-set naturally partitions into a number of coherent subsets, and because the specific characteristics of the form-variants are more discernible to visual perception (and this applies to both asymmetrical and symmetrical form-variants). The discernibility is due to a few key characteristics: (1) nearly all of the "point-line-plane" form-variants have fewer individual components than the circuitous form-variants (each of which has eight lines); (2) many form-variants possess closed shapes ("planes"), which are easier for visual perception to resolve as figures than are lines; (3) there are greater overall contrasts of size and shape both within a given form-variant and among the form-variants, and these contrasts offer the eye more opportunities to compare form-variants for both similarities and differences.

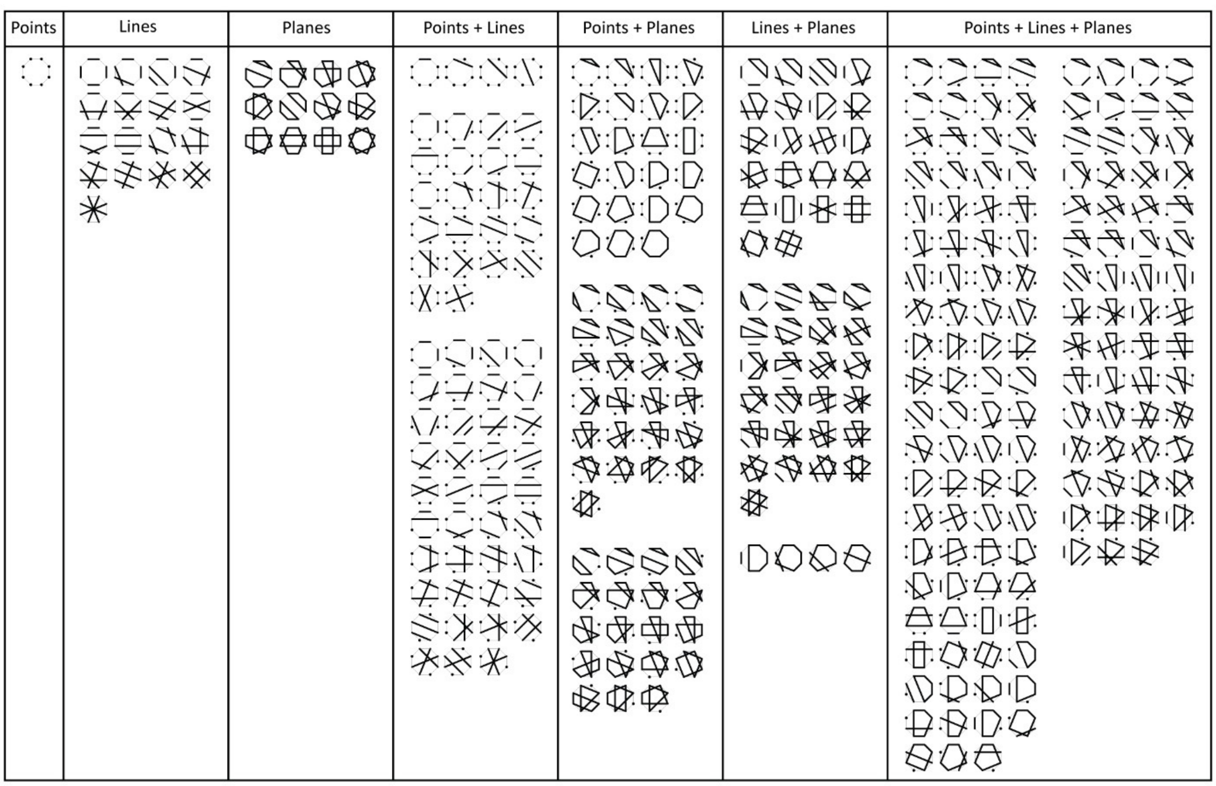

Figure 4: *Form-set of "point-line-plane" permutations of the octagon.*

The resulting "point-line-plane" form-variants are of two types: *singular* and *compound*. Singular types possess only one element: points, or lines, or planes, exclusively. Compound types possess some combination of elements, of which there are four possibilities: (1) points and lines; (2) points and planes; (3) lines and planes; (4) points, lines, and planes. Taken together, these combinations constitute seven subsets of the complete form-set, arranged in the seven column partitions of the table in Figure 4. Most of these seven subsets can be further subdivided into groups with similar features, as for example in the "Points + Lines" subset, which is partitioned into three groups: (1) one line and six points; (2) two lines and four points; and (3) three lines and two points. Combinations, applied within the permutational system, have generated useful subsets and even nested subsets. By comparison to the "circuit-form" variants, these "point-line-plane" variants, both singular and compound types, have proven more adaptable to visual-aesthetic requirements, and so have generated more art works.

The singular-type subset, "Lines" (Figure 4, second column from the left), consists of 17 form-variants, each a distinct arrangement of four lines. This subset is composed in a generally triangular array in the art work, *Peak* (Figure 5). The form-variants of this subset can be grouped in two distinct ways. The forms can be ordered by the number of outer edges of the octagon: one form has four outer edges; one form possesses three outer edges; four forms possess two outer edges; four forms possess one outer edge; and seven forms possess no outer edges. In *Peak*, these five groups are composed, in descending order, in five rows from top to bottom within an equilateral triangle. In addition, the same 17 form-variants can be re-ordered according to the number of axes of reflective symmetry, and these groups have been color-coded in the art work: one form possesses eight axes (yellow); two forms possess four axes (orange); five forms possess two axes (red); eight forms possess one axis (blue), and one form is asymmetrical (dark gray). The five color-coded groups of forms that appear in the art work are arranged in columns in Figure 6. The forms in this art work are as close as any octagon form-set to fulfilling the five permutational characteristics described above.

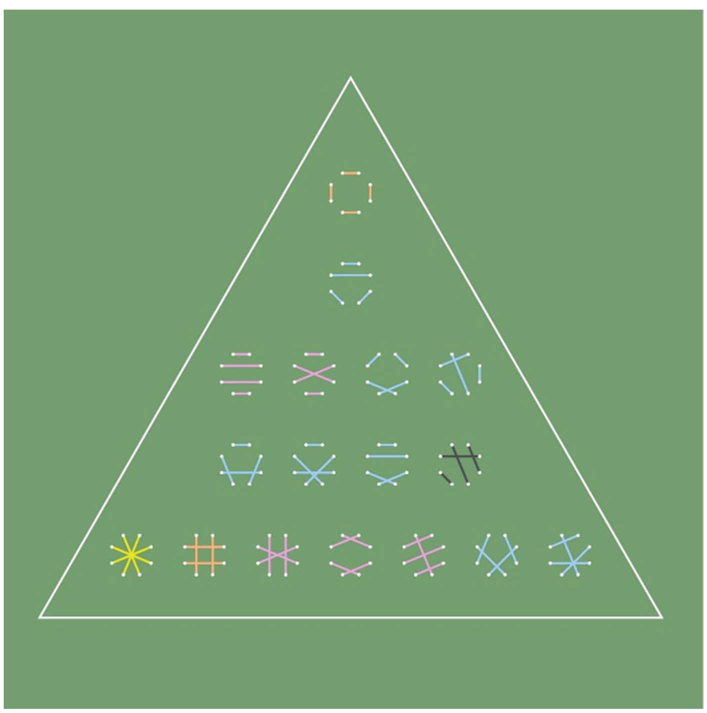

Figure 5: *"Peak" digital print, 2012.*

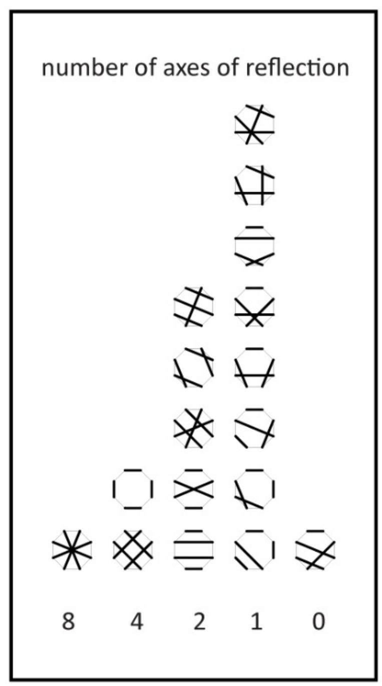

Figure 6: *Form-variant groups indicated by colors in figure 5.*

The compound subset, "Lines + Planes" (Figure 4, sixth column from left) has three further subsets. The art work below, *Zenith* (Figure 7), employs the topmost of these subsets, the 22 forms with two lines and one quadrangle. The forms are arrayed in four rings from the center to the periphery of the compositional circle, in descending order by the number of axes of reflection: the single center form possesses four axes; the next four forms possess two axes; the next ten forms possess one axis; the seven outermost forms are asymmetrical. There are eight different types of quadrangle among the forms. Forms of the same color are rearrangements of two lines and the same quadrangle shape; for example, the square is exclusive to the two yellow forms. These color-groupings, then, extend across different concentric rings. The color-coded groups of forms in the art work are arranged in eight rows in Figure 8.

Figure 7: *"Zenith" digital print, 2014.*

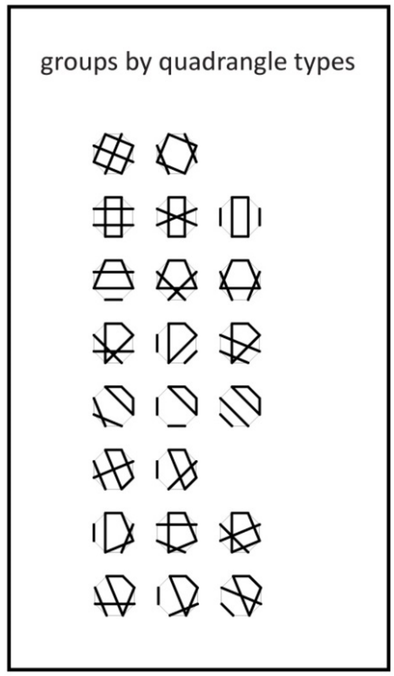

Figure 8: *Form-variant groups indicated by colors in figure 7.*

Conclusion

Form-sets derived from permutations offer more than mathematical "raw material" for composition; the similarities and differences of the form-variants and the inherent completeness of the form-sets constitute aesthetic values in themselves. In this artist's view, permutations are mathematical quantities that yield experiential qualities—of relatedness, integration, and especially wholeness. The form-sets and art works discussed in this paper are but part of an ongoing development in my studio work; the dialectical relationship between mathematics and aesthetics will continue to be at the foundation of that work, acting not in opposition but in reciprocity.

References

[1] J. Mai and D. Zielinski, *Permuting Heaven and Earth: Painted Expressions of Burnside's Theorem*, in Bridges Conference Proc. 2004, eds. R. Sarhangi and C. Séquin, pp. 95-102.

[2] S. Palmer, *Vision Science: Photons to Phenomenology*. Cambridge: The MIT Press, 1999.