Hearing Math and Seeing Music: a Workshop on Pitch Perception and Temperament

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Abstract

Our perception of sound relies on frequencies of waves in the air. Certain frequencies and combinations of frequencies are interpreted by our brains as musical, whether concordant or discordant. In this workshop, we will explore different aspects of mathematics as it relates to pitch perception, including different tuning systems with 12 or more pitches to the octave and the "missing fundamental" effect. We will use both audio and visual aids in our explorations.

1 Introduction

At this point, the relationship between math and music is almost a cliché. Mathematical ideas show up in many guises in music, from rhythmic patterns to voice leading to pitch relationships. This workshop will focus on pitches. The underlying idea is that our ears and brains interpret the frequency of a sound wave as a pitch, with higher pitches having higher frequencies, and ratios of frequencies as intervals. But these ratios and intervals are fundamentally incompatible: there's no such thing as a perfectly-tuned piano. Participants in this hands- (and ears-) on workshop will explore the number theory behind our inability to tune pianos and the compromises that piano tuners and musicians have been making for centuries.

2 Musical and mathematical background

2.1 The problem

In western music, we use a chromatic scale of 12 distinct pitch classes, shown below. Intervals can be described by how many half steps (piano keys) separate the two notes. An octave (a pitch class equivalence) is the distance from C to C, 12 half steps.

When we listen to music, we interpret the frequency ratio between two tones as an interval. A 2:1 frequency ratio is an octave, and a 3:2 frequency ratio is the perfect fifth, another consonant interval. Other intervals, such as the perfect fourth and major third, also have frequency ratios composed of small integers. Somehow, ratios of small integers seem natural to us.

We can generate what is called a Pythagorean scale by choosing a starting frequency and multiplying it by $\frac{3}{2}$ to create the next note, a perfect fifth higher. We continue this process, multiplying each frequency by $\frac{3}{2}$. Mathematically, in a system with 12 pitches to the octave, this corresponds to arithmetic mod 12. We start on pitch class 0 and add 7 for each new note. Because 7 and 12 are relatively prime, we will visit all pitch classes exactly once before getting back to 0. On a piano, the pitch that we get by moving up by fifths twelve times is exactly 7 octaves above the starting pitch. But $\left(\frac{3}{2}\right)^{12} \neq 2^7$, so the Pythagorean scale will not close up to give us perfect fifths and octaves simultaneously. In fact, even if we increased the number of pitch classes, we could never create a scale in which all octaves and fifths are perfect. The two frequency ratios are fundamentally incompatible [2].

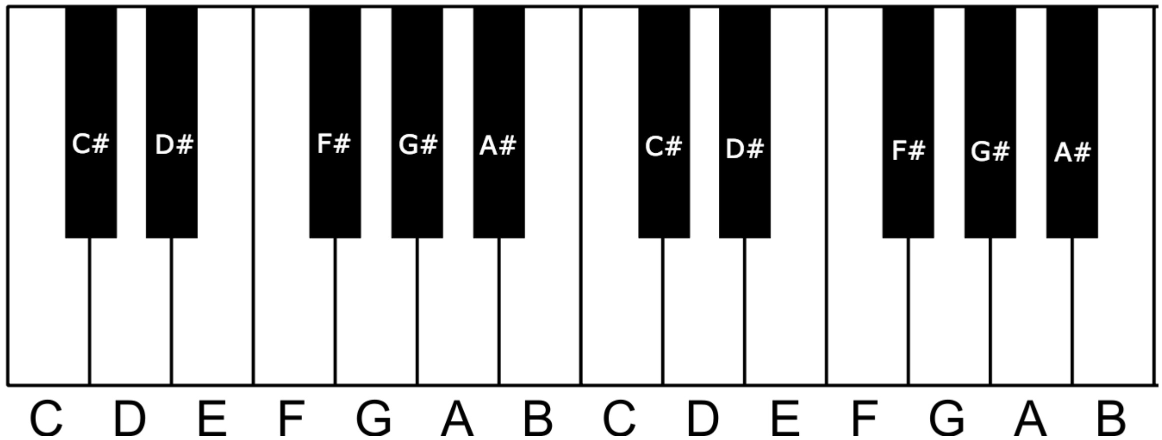

Figure 1 : *A piano keyboard for reference. Perfect fifths (for example, C and G) are 7 half steps apart.*

Other intervals are not compatible either. A "perfect" or just intonation major third (4 half steps on a piano) has a frequency ratio of 5:4, which does not work with either fifths or octaves. For example, on a piano, C-E-G#-C is a sequence of major thirds that ends an octave above where it starts. But if all the major thirds had frequency ratio 5:4, the initial and final C would have a frequency ratio of of $5^3 : 4^3 = 125 : 64 < 2$.

Consequently, every tuning system is a compromise. We want as many intervals as possible to be as close to perfect as possible, and we have a lot of options of how to achieve that.

2.2 The compromises

There's no one right way to tune a piano. Over the centuries, music theorists and composers have used a variety of tuning systems, or temperaments. Early temperaments tended to favor certain keys. For example, the quarter-comma meantone tuning system, popular in 16th and 17th century Europe, compromises by slightly flattening most fifths in order to make thirds more harmonious. The intervals used in this system depend on the starting pitch. In D-based quarter-comma meantone tuning, for example, the fifth from D to A is close to perfect, but the fifth from G# to D# is far too wide. A keyboard tuned in that system will sound beautiful in the key of D but terrible in the key of G#. If a composer didn't care about writing in keys that would use the G# to D# fifth, they wouldn't mind using the temperament.

As composers desired a wider range of harmonic possibilities, equal temperament gradually became the law of the land. In equal temperament, a scale is created by dividing an octave into 12 equal sized half steps with frequency ratio $2^{1/12} : 1$. Octaves are perfect, fifths are slightly flatter than perfect, and major thirds are slightly sharper than perfect. The deviations from perfection are small enough that most listeners do not mind the compromises. Every key is equally in (or out of) tune.

All of the above systems assume a system with 12 chromatic pitches in an octave, but composers and mathematicians from the Pythagoreans to Christiaan Huygens to Harry Partch have also explored temperament systems with other numbers of pitch classes, including 19, 43, and 53 [1]. Perhaps surprisingly, not all increases in the number of pitches in an octave lead to octaves with more perfect intervals. If we want to have good fifths and octaves in an equal temperament, we are asking for numbers that make $\left(\frac{3}{2}\right)^m \approx 2^n$ for integers m and n. Taking the $\log_2$ of both sides, we want fractions $\frac{n}{m}$ that are close to $\log_2\left(\frac{3}{2}\right)$. Thus, the denominators of the convergents of the continued fraction for $\log_2\left(\frac{3}{2}\right)$ show us good candidates for equal temperament systems. (It happens that $\frac{7}{12}$ is a convergent for $\log_2\left(\frac{3}{2}\right)$, corresponding to the fact that 7 half steps in 12-tone equal temperament is close to a perfect fifth.) If we wish to approximate other intervals well, we can perform similar computations.

2.3 Playing with pitch perception

In discussions of temperament, we sometimes pretend that a note on a piano is a pure sine wave with a particular frequency, but that is a simplification of the truth. Musical instruments and voices do not produce perfect sine waves. They produce waves that are combinations of a fundamental frequency and harmonics that are integer multiples of the fundamental, and the amplitudes of the harmonics help determine the timbre, or tone quality, of a sound.

Because we need to be able to distinguish pitches even when some information is missing, we can fall prey to an auditory illusion called the "missing fundamental." For example, if we created a wave composed of sine waves with frequencies 200, 400, 600, and 700 Hz, we might hear a pitch with frequency 100 Hz. Our brains notice that the frequency profile doesn't fit the typical output of a sound that is 200 Hz, so it "corrects" the sound for us. We hear the frequency that is the greatest common factor of the frequencies that are actually present. This effect is exploited in telephones and some musical instruments, such as the organ, allowing us to hear frequencies that those instruments cannot produce.

3 The Workshop

The workshop will have two main aspects, aural and visual. Aurally, we will use the free audio program Audacity to generate waves with the desired shapes and frequencies; visually, we will use Audacity and Desmos to visualize wave combinations. Participants will be gently guided by an open-ended worksheet and the workshop leader.

After a brief introduction to Audacity, we will begin by listening to intervals with frequency ratios $2:1$, $3:2$, and $5:4$, which correspond to the octave, perfect fifth, and just intonation major third. We can then use only $3:2$ and $2:1$ frequency ratios to generate pitches in a 12-tone Pythagorean scale, comparing the Pythagorean major third to the just intonation major third. We will hear beats if we play the two chords together and can use either Audacity or Desmos to see why they occur when intervals are nearly, but not quite, perfect and when notes are nearly, but not quite, the same.

When we create 12 perfect fifths with frequency ratio $3:2$ (dividing by 2 when necessary to adjust octaves), the piano says we should be exactly an octave away from our starting pitch, but we are not. We will listen to the interval, called the Pythagorean comma, between the pitch we got by using fifths and the pitch we got by using octaves. Participants who wish may look at several historical tuning systems and how they dealt with the Pythagorean comma.

12-tone equal temperament eventually won the day. We will figure out how to generate a 12-tone equal temperament scale and compare the intervals we get with the ones from Pythagorean, just, and other tuning systems. We will also explore tuning systems with more than 12 pitch classes.

To play with pitch perception, we will listen to and visualize the missing fundamental effect. Participants will generate sine waves with frequency relationships such as $2:3:4$. This frequency pattern doesn't match the overtone series for the lowest sine wave present, so we will hear a note an octave below the lowest one present. We will test the limits of the missing fundamental by seeing how few overtones we can have and still hear the effect.

When we listen to the missing fundamental, we will notice that the timbre of the tones changes as we change the frequencies present. We will explore timbre by playing with the amplitude (loudness) of overtone waves we include in a tone.

Participants will have ample opportunities to pursue topics that particularly interest them and exchange ideas about how to use material from the workshop in either their classrooms or their own creative endeavors.

Familiarity with the layout of a piano keyboard or basic knowledge of music notation will be helpful but not necessary. It will also be helpful to bring headphones and a laptop with Audacity already installed on it, but some Audacity-equipped computers and headphones will be available for participants who don't bring computers.

References

[1] D. Benson, *Music: A Mathematical Offering* (2007), Cambridge University Press.

[2] S. Isacoff, *Temperament: The Idea that Solved Music's Greatest Riddle* (2001), Alfred A. Knopf.