Schematic Drawings of the Polychora

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Abstract

Regular 4-dimensional polychora are often shown as still images on a planar medium. This paper deals with the possibility of representing the edge-graphs of the polychora with schematic drawings instead of geometrically accurate projections. A set of simple visual principles produces symmetric and visually appealing illustrations of the 5-cell, the hypercube and the 16-cell.

Introduction

Just as 3-dimensional polyhedra can be shown on a plane by projecting them, the 4-dimensional polychora can be shown in our Euclidean 3-space as 3-dimensional projections [2]. Of the different projection methods, two are more common: orthographic and stereographic projection. Thus, these two methods widely appear in physical models of projected polychora. Orthographic projections surface in the form of various ball-and-stick designs, such as *Zometool* constructions [3], while stereographic projections show up e.g. as 3D-prints [5]. In addition to physical models, there are numerous computer-generated animations of stereographic projections, like the film *Dimensions* [1], and software applications that allow virtual manipulation of the projection outcome by the user (e.g. *Jenn3D* [4]). Although these visualization methods are the most instructive, there remain many situations where we have to settle for a still image.

When the projections of the polychora are portrayed on a page of a book or on the computer screen as still images, they are usually photographs of physical models or rendered images from 3D modeling software. Although we can choose an appropriate viewing angle, distance and focal length of the camera, the elements of complex edge-graphs often occlude each other from view in the planar composition. Because the achievement of pleasant planar layout is not possible just by adjusting the above-mentioned parameters, I will describe here a method of producing illustrations that are schematic rather than geometrically correct, and will use this method for the representation of the 5-cell, the hypercube and the 16-cell.

Methods

When designing visualizations of the polychora, often only the vertices and edges are drawn, and the shape of the faces and the cells is inferred on the basis of this information. Also the illustrations described here show only the edge-graphs of the polychora. The examples of schematic drawings represent the stereographic projections of the simplest three of the regular convex polychora (Figure 1):

- The 5-cell, composed of 5 vertices, 10 edges, 10 faces and 5 cells
- The hypercube, composed of 16 vertices, 32 edges, 24 faces and 8 cells
- The 16-cell, composed of 8 vertices, 24 edges, 32 faces and 16 cells

Since the 4-dimensional polychora are already, by necessity, much distorted when represented on a plane, I propose an approach that favors the planar symmetry and clarity of the composition at the expense of geometric accuracy and illusion of depth. Instead of applying parallel or perspective projection to flatten the 3-dimensional result of the first projection onto a plane, I use iterations of free-hand drawing to arrive at visually appealing layouts. This approach is realized by following a set of simple local and global principles:

- The edges meeting at a vertex should be evenly spaced (in the angular sense) around it.
- At the crossings, the edges should meet each other as perpendicularly as possible.
- Where possible, all the parts of the polychoron should be portrayed at the same scale and with even proportions.
- The overall 2-dimensional layout of the drawing should be as symmetric as possible

As it is evident that not all of the principles can be followed throughout, the right balance of the trade-offs is found after several sketches. This method corresponds to the tradition of pictorial composition in visual arts.

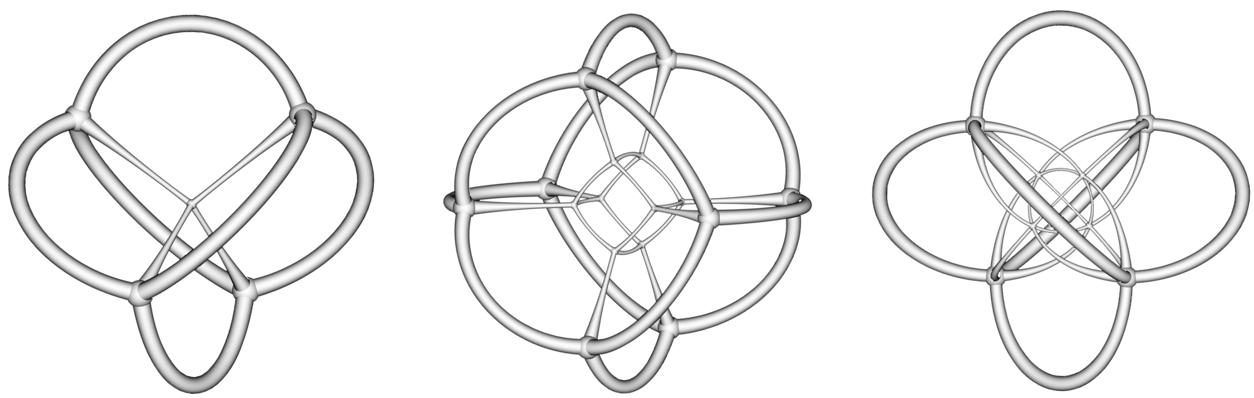

Figure 1: *Stereographic projections of the 5-cell, the 8-cell (hypercube) and the 16-cell, respectively (Initial images exported from* Jenn3D *software)*

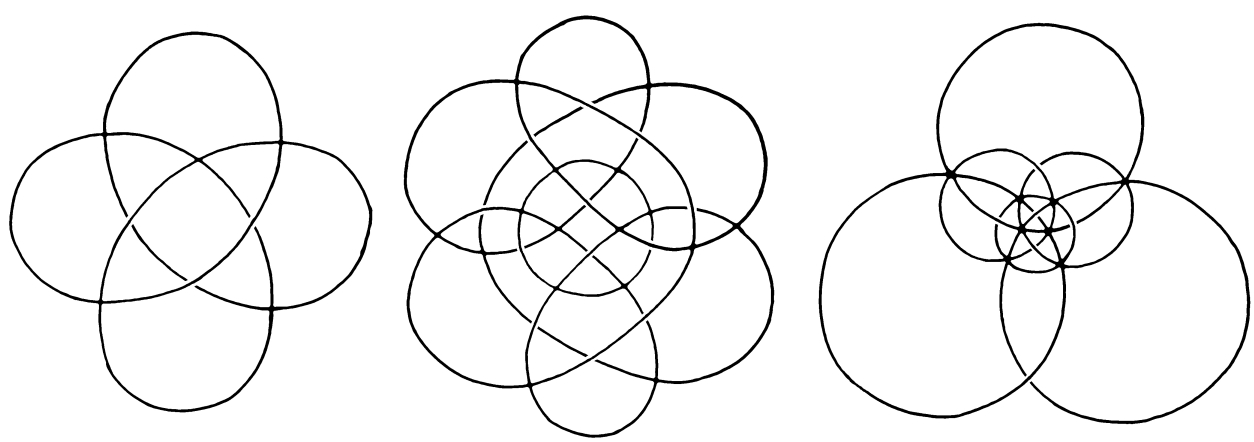

Figure 2: *Sketches for the new layouts of the edge-graphs*

If we ignore the crossings, the new layout of the 5-cell has 4-fold rotational symmetry and resembles a knot or a stereographic projection of the square antiprism (Figure 2, left). The edge-graph of the hypercube is seen to be composed of a circle and more complex, knot-like curve with bilateral symmetry (Figure 2, center). The 16-cell is a special case, since its cell-centered stereographic projection can be again stereographically projected without the edges blocking each other from view. Consequently, the new layout has the shape of the stereographic projection of tetrakis hexahedron, a Catalan solid composed of

24 isosceles triangles (Figure 2, right). The center of the layout is slightly blown out to even the proportions of the triangles in the final drawing of the 16-cell (Figure 3, bottom).

Besides the layout, my designs deviate a bit from the norm in their detail and finishing. Instead of the usual spheres and cylinders, the vertices and edges are given just a slight, ribbon-like thickness to represent overlapping. At the vertices, the edges simply join smoothly to give immediate visual distinction between vertices and crossings. To create a stronger illusion of depth, the design is drawn in atmospheric perspective.

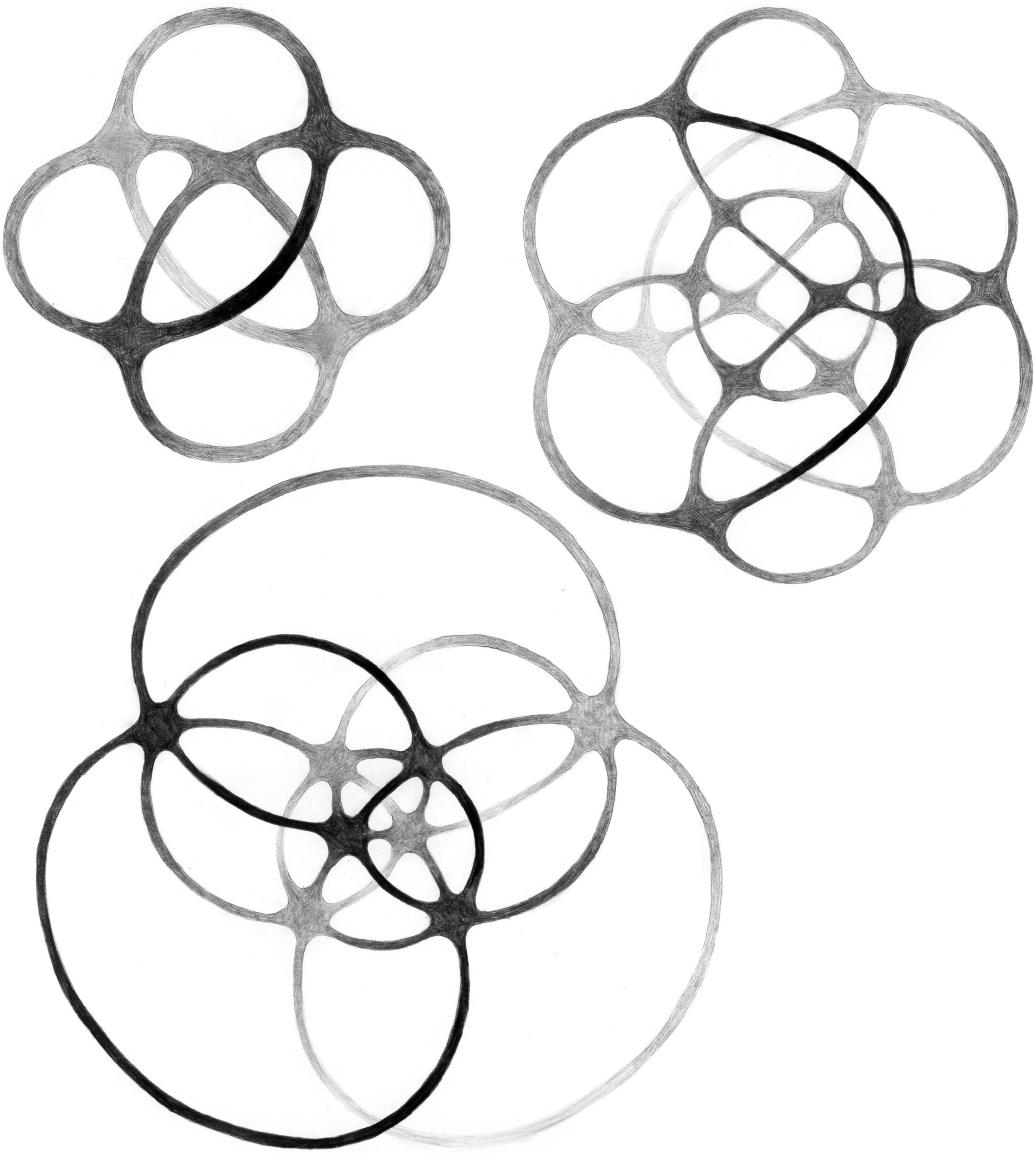

Figure 3: *Free-hand drawings of the 5-cell (upper left), the 8-cell (hypercube) (upper right) and the16-cell (bottom)*

Results

The final results—free-hand drawings of the 5-cell, the hypercube and the 16-cell—are shown in Figure 3. All the parts are clearly visible in the free-hand drawings, and it would be easy to label the vertices and edges with, e.g., names, colors, or numbers.

Finding appropriate places for labels referring to the faces or to the cells is not so straightforward. Remembering the two-way vertex-to-cell and edge-to-face duality from the hypercube to the 16-cell, and from the 5-cell to itself, one can use a drawing of the dual polychoron to refer to the 2- and 3-dimensional elements of the polychoron in question.

Conclusion

Like the cubist painters who used their artistic freedom to distort the overall shape of their subject matter to combine different views of the visually important elements, the schematic drawings of the polychora favor the planar clarity of the representation at the expense of geometric accuracy. Their visual appeal remains of course a matter of subjective taste. Personally, I enjoy the oscillation between the three interpretations of the drawings—the planar, the spatial and the hyperspatial ones. Although this drawing method could be used to represent the edge-graphs of the more complex of the remaining polychora, the increasing number of the vertices and edges would doubtlessly impair the spatial interpretation even more than in the examples shown here.

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