From Mathematical Curves to Decorative Ornaments

Susan McBurney
Western Springs, Illinois 60558
E-mail: smcburney108@gmail.com

Abstract

Delicate, precise and sometimes mathematical designs have long been used not only for decorative purposes, but also as a deterrent to counterfeiting on bank notes, stock certificates and other valuable documents. The intricacy of the designs is always appealing to mathematicians, and also raises the question of how they are created. In some cases, they are the result of beautifully crafted machinery; in others they are mathematically defined and executed with modern graphic software programs. Not only does this inspection lead to an appreciation of past methods and workmanship, but it also raises the question of how improved designs might be created. This paper will briefly explore these issues.

Mathematical Definition

The curve traced by a point P on a moving circle M rolling on the *outside* of a fixed circle F is called an epicycloid. Albrecht Dürer studied these in 1525. If the circle M is rolling *inside* circle F, the resulting curve is a hypocycloid. Variations arise when the point P is in or outside the rolling circle. (Figure 1).

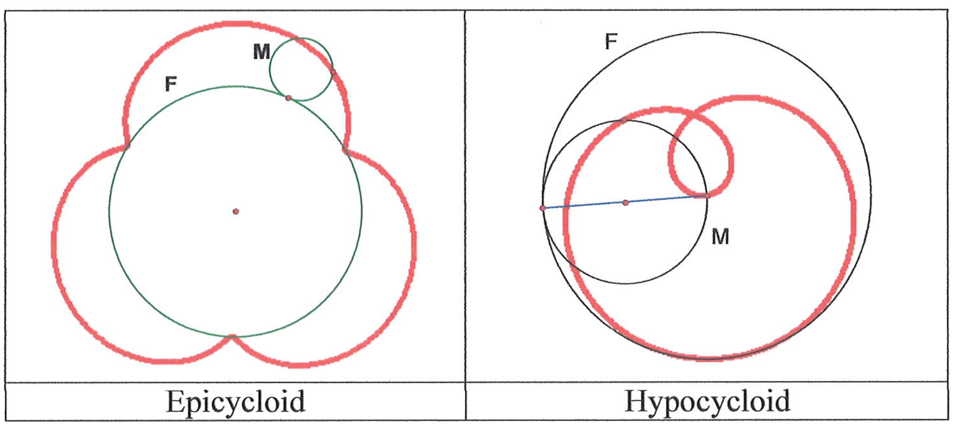

Figure 1

Parametric equations for the hypocycloid are not too difficult to derive [1]. They are:

$$x = (R - r)\cos\theta + r\cos[(R - r)/r]\,\theta,\quad y = (R - r)\sin\theta - r\sin[(R - r)/r]\,\theta$$

where R and r are the radii of the fixed and rotating circle, respectively. Analogous derivation yields equations for the epicycloids. [1]

Geometer's Sketchpad Program

Aaron Dunigan AtLee has written a Geometer's Sketchpad spirograph program which simulates the familiar plastic toy. Figure 2 shows the effect of moving the point P closer to the edge of the interior wheel (away from the center) while generating hypotrochoids. A hypotrochoid is the curve created when the trace point is a distance d from the center of the rotating circle.

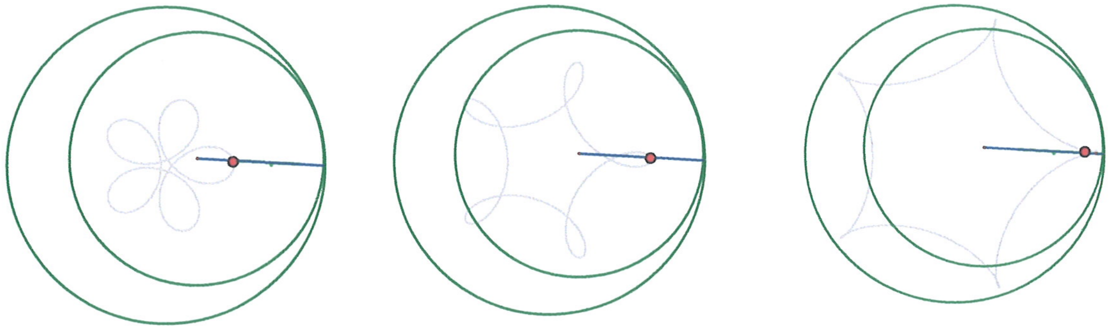

Figure 2: *Moving the trace point towards the edge of the wheel.*

Figure 3 shows the effect of keeping the hole position the same, but decreasing the wheel size. When the point P moves outside of the wheel, the generated curve becomes an (extended) hypocycloid.

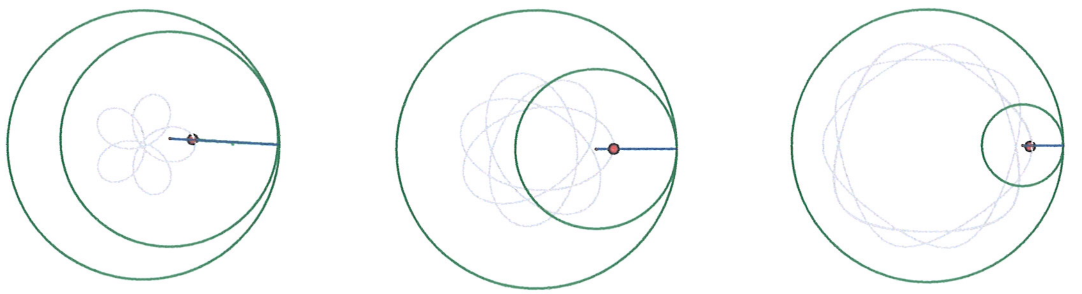

Figure 3: *Trace point in the same relative position on a shrinking wheel.*

Gear Shapes

Changing the *shape* of the rotating generator adds new designs to the collection. For instance, using a Reuleaux triangle (shown on the left in Figure 4) rotating within a circle, generates the designs on the right. A Reuleaux triangle is a curve of constant width formed by drawing arcs from each corner of an equiangular triangle to connect the two opposite corners.

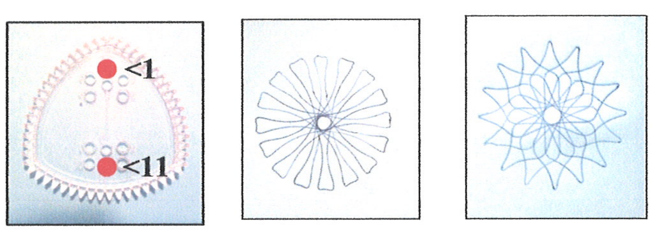

Figure 4: *A Reuleaux triangle gear. (b,c) Patterns generated by Hole 1 and Hole 11.*

The "football" gear, made from two arcs reflected about a central line, gives the designs shown in Figure 5. Again, a small displacement gives a much different output. The gear has 32 indentations on each side, and it is rolling inside a fixed circle with 96 "cogs".

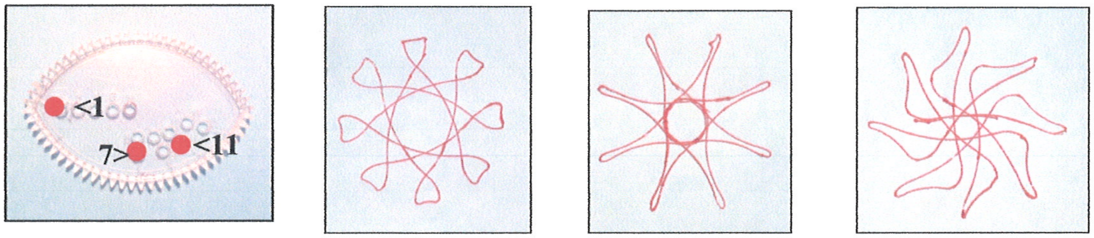

Figure 5: *"Football" gear. (b,c,d) Patterns generated by Hole 1, Hole 7, Hole 11.*

A third gear shape is shown in Figure 6. This generates the rectangular hyperbola curve formed by a point P on a square/rectangle. Initially a square is situated with the lower left corner at the (0,0) mark on a Cartesian grid. Then the upper right corner P is traced as all the rectangles with the same *area* as the original square are formed, both to the right and the left. The generated curve is shown to the right. The beautifully simple formula for this curve is $x*y = k^2$, a constant [3].

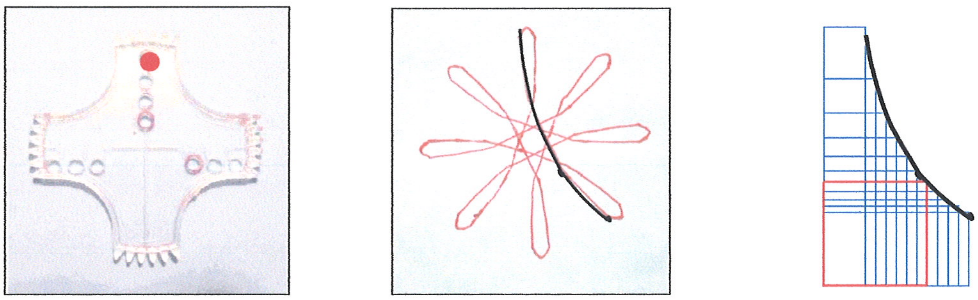

Figure 6: *Design drawn by gear shown. Hole 1. Rectangular hyperbola..*

Spiral Program

Another interesting application for generating curves is the Spiral Program, an app written for certain tablet devices. It draws *straight lines* of first increasing and then decreasing length in a spiral formation between the starting point and the edge of the outside circle. These form very intricate designs as shown in Figure 7. Notice the designs are made only of straight lines, as shown in the enlargement to the right.

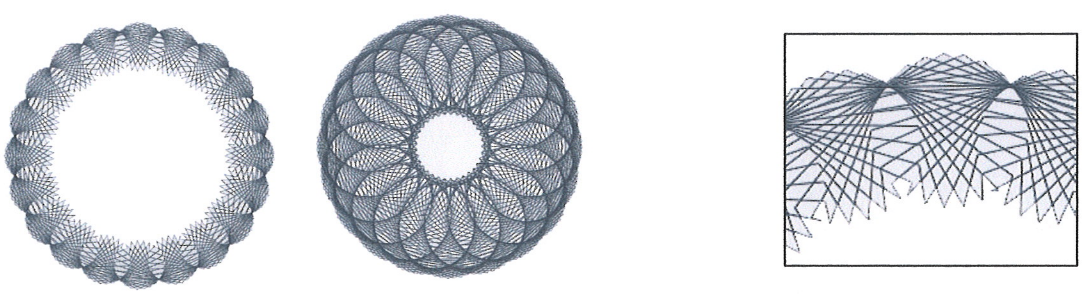

Figure 7: *Designs drawn by the Spiral Program with different starting points.*

Early Machines for Generating Designs

With the power of modern software it is possible to explore realms of decoration that were not possible before. This in no way diminishes the achievements of pre-computer devices. Many were marvels of expertise that reflected existing technology. The first spirograph (not capitalized) was invented by P. H. Desvignes in Vienna in 1848. It is pictured in Figure 8 with some of the spiral designs it could create [2]. Three variables contributed to these designs: the rotation of the main table, the linear motion of the pen carriage away from the central wheel, and a slight uniform rotation of the pen around the center of its carriage.

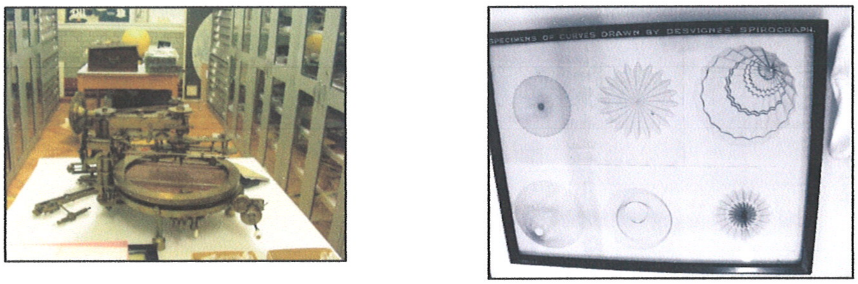

Figure 8: *The first spirograph and designs.*

The refinement of metallurgy that made possible precise gear manufacturing, for example, led to more accurate devices in the early 19th century that created new designs. And, in each case the actual manufacture was preceded by clever innovators who anticipated what might, in theory, be possible. Figure 9 shows the geometric chuck for generating epicycloids. Each stage could vary in speed of rotation center of rotation offset. This particular machine was constructed in 1906 [2].

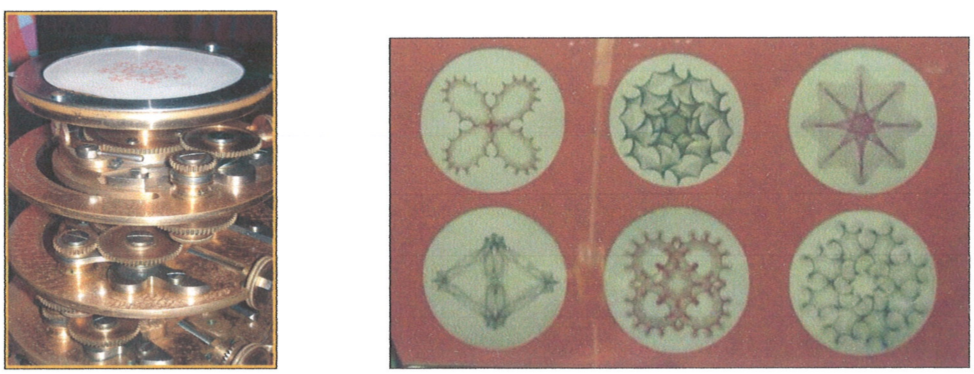

Figure 9: *The four-level geometric chuck and designs.*

Modern computer programs can now move beyond the physical restraints of previous eras. Another area of exploration would be to modify the *path* of the generator. But again, thanks to mathematicians from the past, the theory of curves has been well developed such that later technological advances can be exploited to create new designs.

References

[1] Eli Maor, *Trigonometric Delights,* Princeton University Press, 1998
[2] Photos by author, *Archives*, London Science Museum, ©2006
[3] C. Alsina and R. Nelsen, *Icons of Mathematics*, Dolciani Mathematical Expositions #45, MAA 2011