Fractal Tiling Illustrations of Geometric Series

Lorelei Koss
Dickinson College
Department of Mathematics and Computer Science
P.O. Box 1773
Carlisle PA 17013
koss@dickinson.edu

Abstract

A *proof without words* is a style of proof that demonstrates a mathematical identity through the use of pictures instead of text. In this paper, we discuss how to use fractal tilings to illustrate sums of geometric series.

Introduction

This paper explores the intersection between tilings and fractals, two mathematical topics having obvious connections with artistic ideas, and geometric series, a mathematical idea that is completely abstract. The use of tilings in art and architecture dates back to ancient civilizations, and examples from Sumerian, Greek, Roman, Islamic, and Indian cultures abound [4]. With the advent of computer graphics in the 1970s, we have come to understand that tiles can have fractal boundaries. Here, we use fractal tilings to illustrate sums of geometric series.

A *plane tiling* is a countable family of closed topological disks which cover the plane without gaps and overlapping only along their boundary. For example, a square can be used to tile the entire plane in a checkerboard pattern. In this paper, we are interested in tiles that are themselves formed by some number of smaller copies of themselves. Formally, a *rep-k-tile* T is a tile T that can be split into k congruent parts, each of which is similar to T. For example, a square is a *rep-4-tile* because it can be divided into four smaller squares. A pentagon is not a rep-n-tile because it cannot be divided into n smaller copies of itself for any number n. Here, we are interested in tiles that have a fractal boundary, such as the rep-5-tile shown in Figure 1. While we won't give a formal definition of a fractal, we note that fractals often exhibit self-similarity and a detailed structure at all scales.

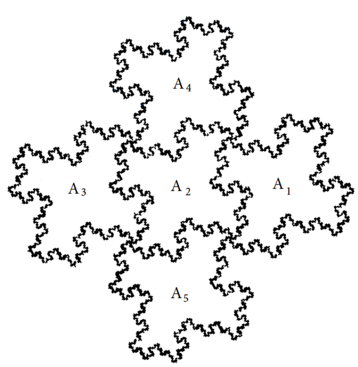

Figure 1 : *A rep-5-tile*

Greek mathematicians understood geometric series. Both Archimedes and Euclid were able to find the sums of series such as

$$a + ar + ar^2 + ar^3 + \dots \tag{1}$$

where a and r are real numbers. To find the formula for the sum, let $s_n = a + ar + ar^2 + \ldots + ar^{n-1}$ denote the sum of the first n terms of the series. Multiplying both sides of the equation by r, we obtain $rs_n = ar + ar^2 + \ldots + ar^n$. Subtracting the two equations gives

$$s_n - rs_n = (a + ar + ar^2 + \ldots + ar^{n-1}) - (ar + ar^2 + \ldots + ar^n) = a(1 - r^n).$$

Simplifying, $s_n = a(1-r^n)/(1-r)$. Then the sum of the series is the limit as n gets large, and $\lim_{n\to\infty} s_n = a/(1-r)$ if $|r| < 1$. Therefore, if $|r| < 1$, the sum of the geometric series in Equation 1 is $a/(1-r)$. For example, the geometric series with $r = 1/3$ and $a = 1$,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.$$

This result is not intuitive, though, as it relies on algebraic manipulations and a formal understanding of limits. The purpose of this paper is the provide a visualization of sums of geometric series. These types of proofs are commonly known as "proofs without words," and there are entire books proving various mathematical results using this technique [6, 7]. In fact, examples of visual proofs about geometric series appear in [6, 7], but these examples all involve polygonal shapes; the innovation of this paper is to use tiles with fractal boundaries.

The Mathematics

The construction of fractal rep-n-tiles is described in [1] and explained at a more basic level in [3]. It involves finding an inverse of a linear transformation and an appropriate set of translations. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix where $a, b, c,$ and d are integers. We say that M is *expansive* if all eigenvalues of M have modulus greater than 1. The action of M on a region in the plane can be thought of as a sequence of reflections, expansions, compressions, and shears. We also need a set of vectors that represent the necessary translations of the smaller sub-tiles to each other. These vectors are formally known as a *complete residue system*, and they consist of a complete set of coset representatives for the quotient group $\mathbb{Z}^2/M\mathbb{Z}^2$. The following theorem from [1] ensures that we can construct rep-n-tiles from the matrix M and a complete residue system.

Theorem 1. *Let M be an expansive matrix with $n = |det(M)|$, and let $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a complete residue system for M. Define $f_j(\mathbf{z}) = \mathbf{y}_j + M^{-1}\mathbf{z}$ for $j = 1...n$. Then there is a unique rep-n-tile A which is the union of n tiles A_j that have disjoint interiors and satisfy $A_j = f_j(A)$.*

As an example, Figure 1 shows the rep-5-tile that results from the matrix $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, using the complete residue system consisting of the vectors $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{y}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{y}_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The sub-tiles A_j for $j = 1...5$ are labeled, and the entire tile A satisfies $A_j = f_j(A)$ for each j.

We can use the functions f_j described in Theorem 1 to produce a coloring of the rep-n-tile A that illustrates the sum of a geometric series. Any tile A constructed in such a manner has area n. Begin with tile $A_1 = f_1(A)$, which has area 1. The compositions $f_2(f_j(A))$ for $j = 1...n$ divide the tile A_2 into n smaller tiles, each of area $1/n$. Define $A_{21} = f_2(f_1(A))$. In general, let $A_{2^k1} = f_2^k(f_1(A))$, where f^k denotes function composition. By induction, the area of each region A_{2^k1} is equal to $1/n^k$. Coloring only the set of tiles $\{A_1, A_{21}, A_{221}, \ldots\}$ results in a region that has area

$$1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots = \frac{n}{n-1},$$

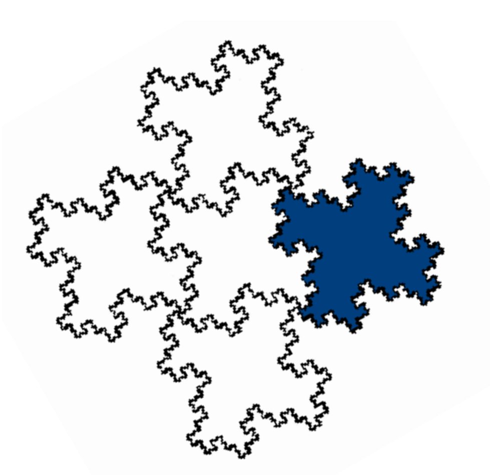

Figure 2 : *Dark blue region has area 1*

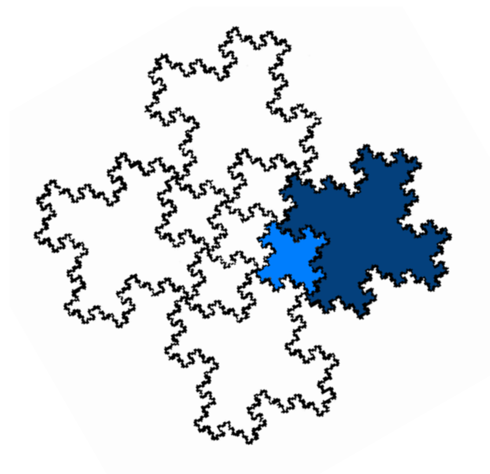

Figure 3 : *Colored area is* $1 + \frac{1}{5}$

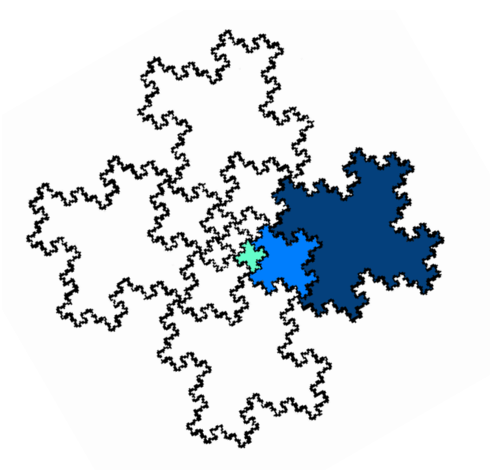

Figure 4 : *Colored area is* $1 + \frac{1}{5} + \frac{1}{5^2}$

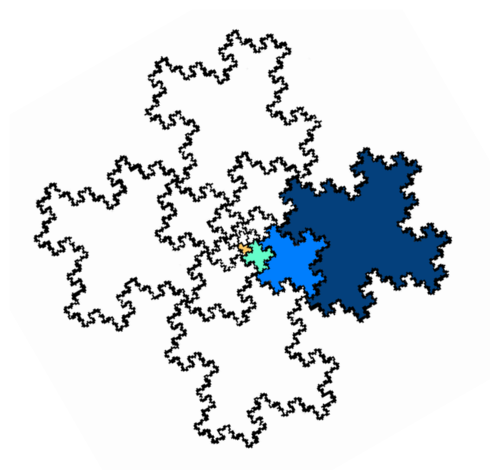

Figure 5 : *Colored area is* $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$

by the formula for the sum of a geometric series. To illustrate the sum of an arbitrary geometric series, we simply scale the original rep-n-tile by a factor of a. In this case, the area of each region $A_{2^k 1}$ is equal to a/n^k, and we end up with a picture with colored area

$$a + \frac{a}{n} + \frac{a}{n^2} + \frac{a}{n^3} + \ldots = \frac{an}{n-1}.$$

The tiles constructed in Theorem 1 may not be topological disks. While the procedure described above will still result in an illustration of a geometric series, for the purposes of this paper it is easiest to visualize the sum of the series when the tiles are topological disks. Necessary and sufficient conditions for ensuring that rep-n-tiles tiles are topological disks can be found in [2]. We note that [5] explains other approaches to constructing rep-n-tilings.

We provide two examples of visualizing the sum of a geometric series. Figures 2 through 5 illustrate this process with the fractal rep-5-tile A with area 5 shown in Figure 1. Figure 2 shows the sub-tile A_1 in dark blue, which is a region with area 1. In Figure 3, we divide the sub-tile A_2 into five smaller pieces and use light blue to color the region A_{21}, which has area $1/5$. Figure 4 shows the region A_{221} with area $1/5^2$

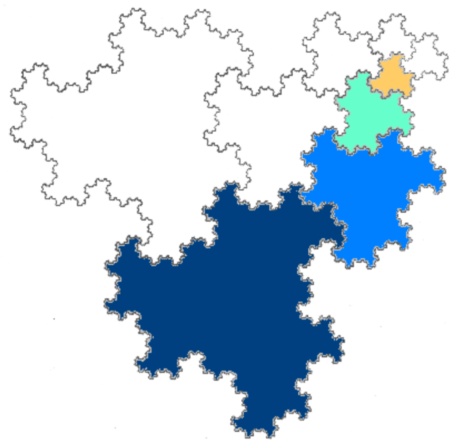

Figure 6 : *Colored area is* $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3}$

colored green, and Figure 5 shows the region A_{2221} of area $1/5^3$ in orange. Continuing this process, we see by symmetry that we will eventually color $1/4$ of the entire region A. That is, at each step, for every colored fractal sub-tile there are four identical un-colored sub-tiles of the same shape. One of the uncolored ones is still subject to further subdivision and eventually will get infinitely small. Since the original tile A has area 5, this illustrates that the sum of the geometric series

$$1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots = \frac{1}{4}(\text{area } A) = \frac{5}{4}.$$

For a second example, we only show the fourth step of the process. Figure 6 shows a fractal rep-3-tile A with area 3. We color one sub-tile of area 1 dark blue, a sub-tile of area $1/3$ light blue, a sub-tile of area $1/3^2$ turquoise, and a sub-tile of area $1/3^3$ orange. Continuing this process, we see that we will eventually color $1/2$ of the entire region A, illustrating that the sum of the geometric series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots = \frac{1}{2}(\text{area } A) = \frac{3}{2}.$$

References

[1] Christoph Bandt. Self-similar sets 5. integer matrices and fractal tilings of R^n. *Proceedings of the American Mathematical Society*, pages 549–562, 1991.

[2] Christoph Bandt and YANG Wang. Disk-like self-affine tiles in R^2. *Discrete & Computational Geometry*, 26(4):591–601, 2001.

[3] Richard Darst, Judith Palagallo, and Thomas Price. Fractal tilings in the plane. *Mathematics Magazine*, 71(1):12–23, 1998.

[4] Owen Jones. *The grammar of ornament*. B. Quaritch, 1868.

[5] Peng-Jen Lai. How to make fractal tilings and fractal reptiles. *Fractals*, 17(04):493–504, 2009.

[6] Roger B Nelsen. Proofs without words: Exercises in visual thinking. *The Mathematical Association of America*, 1993.

[7] Roger B Nelsen. Proofs without words II: More exercises in visual thinking. *The Mathematical Association of America*, 2001.