Cayley Cubic and the Visual Arts

Jean Constant
Hermay.org
2300 South Ct.
Santa Fe, NM, 87505, USA
E-mail: jconstant@hermay.org

Abstract

It puzzles many that cubic surfaces, discovered and classified more than a hundred years ago, are still very present in mathematical studies today. This presentation briefly reviews the theory and principle of these particular surfaces and submits that recent developments in computer-aided technology may be consequential in the renewed interest of scientists, engineers as well as artists for this area of study. Using recent sophisticated mathematical visualization programs, I investigate the Cayley cubic and explore its surface in a visual art context to highlight the close connectivity between pure mathematics and our larger aesthetic environment.

Introduction

The Cayley cubic in its original form has been the object of many studies since it was first presented to the Royal Society of London in 1869. An impressive amount of papers continues to be published today on cubic surfaces and Del Mezzo surfaces. In the word of A. Henderson "While it is doubtless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year to year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of the discovery of the twenty-seven lines upon the cubic surface." [1].

The appeal of the problem relating to cubic surface may be dual. First it calls for challenging and elegant demonstrations of mathematical reasoning and second the recent development in computer-aided technology allows for more exact and precise projection than the plaster models that were used at the turn of the nineteen century to demonstrate those projections. In the following presentation, I focus primarily on the Cayley cubic surface and fast-forward to our modern environment to explore the surface with a mathematical visualization program called SURFER to look at the outcome from a graphic and aesthetic perspective.

The Cayley cubic

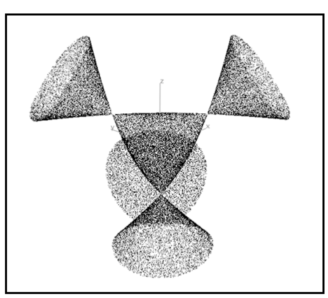

Figure 1: *3D-XplorMath –The Cayley cubic*

Arthur Cayley was a 19th century British mathematician who produced a huge body of work. He published over 900 papers and more than fifty concepts and theorems of mathematics that covered almost every aspect of modern mathematics. He united projective geometry and metrical geometry, which is dependent on sizes of angles and lengths of lines [2].

In algebraic geometry, the Cayley surface is a unique cubic surface in 3-dimensional projective space with four conical, or ordinary double points, the maximum possible for cubic surface (Fig.1). Arthur Cayley and mathematician George Salmon collaborated on defining the maximum finite number a cubic surface contains. They wrote a significant paper demonstrating that, based on Salmon' proof relating to the invariant of a cubic, the number of lines in this type of surface must be equal to 27, and that 45 tri-tangent planes, represented by three mutually intersecting lines, intersect the surface. The discovery of the 27 lines on a cubic surface identify the first non-trivial result on algebraic surfaces of order higher than 2 [3].

In graphical or projective perspective, parallel lines intersect at a vanishing point [4]. Cubic surfaces are implicit, polynomial surfaces in projective three-space with terms of degree three or less [5]. Swiss mathematician L. Schlafli was the first to classify the various types of singularities following the 27 points Cayley-Salmon cubic surface theorem [6]. In 1871, Clebsch gave a model of a cubic surface, called the Clebsch diagonal surface, where all the 27 lines are defined over a set field and can be identified with objects or vectors arising in representation theory.

Interest in this very abstract topic declined until computer-aided technology allowed the transfer of mathematical calculation into precise, effective visualization. Holt-Netravali and Mundy-Zisserman pointed out in that computer vision, it is essential to derive properties of curves and surfaces that are invariant to perspective projection and to be able to compute these invariants reliably from perspective image intensity data [7].

The SURFER program

The SURFER program was developed to respond to these specifications. It has become an invaluable tool in the digital environment for scholars, educator, artists and all that do not have the technical background and expertise of mathematicians and engineers to convert pure mathematics into beautiful visualizations.

Surfer, a Java-based extension of the program SURFER2008 was developed as joint project of the MFO (Mathematisches Forschungsinstitut Oberwolfach) and the Technical University of Kaiserslautern [8]. It is today part of the MFO-IMAGINARY traveling exhibit, a very successful program of promotion of Mathematics.

Its origin can be found in the programs Surf and Spicy that were projects of the research group Algebraic Geometry at the Department of Mathematics of the Johannes Gutenberg-Universität at Mainz (Germany) under the direction of Stephan Endrass [9]. Surf was designed to visualize algebraic curves and surfaces by writing scripts to allow the program to execute them in the user interface. The program was designed to work with one default equation and allow the user to alter all parameters until a final image could be completed. Each high-resolution surface and curves can be saved and imported into most existing graphic applications.

Cayley cubic and SURFER

The visualization of the Cayley cubic in the SURFER program is initially set to a default equation designed by German mathematician Felix Klein:

(b-0.5)*(x+y+z-1)^2+(a-0.5)*(x+y+z+3)^2+x^3+y^3+z^3+1-0.25*(x+y+z+1)^3=0

My interest in using it was to generate interesting shapes using several variations the program was allowing (Fig. 2).

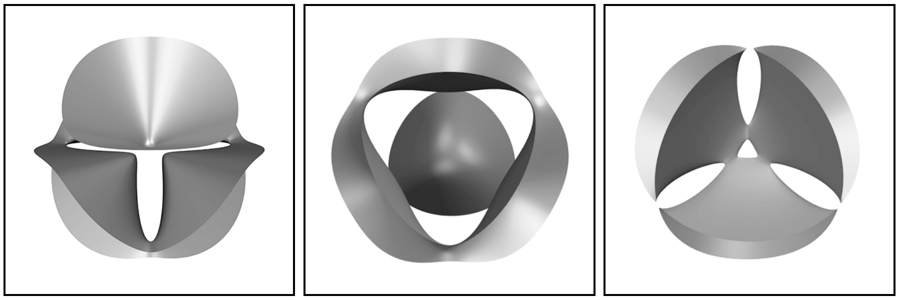

Figure 2: *SURFER – Cayley cubic variation*

The program draws algebraic surfaces produced by simple equations. The shape can be rotated, reduced or enlarged and there is a possibility to change the object colors. I did not use the color tool at the early stage wanting to concentrate on the object shape and appearance. The first challenge I had was to extract a coherent figure that would look good on a 2D surface. There are elements of perspective and symmetry in the projection onto a plane that require careful positioning to convey properly the particular dynamic and balance of a two-dimensional surface representation.

I transferred the shape I selected into a graphic program that had more sophisticated tools to deal with the specific of graphics composition [10]. I added density to the pixels in some areas to increase the perception of depth, created additional shadows and added more nuanced colors to bring up a form both mathematically correct and that could relate to an abstract sculpture or an antique artifact of some kind, which is not surprising as many ancient cultures intuitively were using geometry to define shapes and express perfection (Fig. 3).

Figure 3: *The Minotaur. Minoan figurehead*

Conclusion

This project highlighted several points that are very relevant to collaboration between Mathematics and the visual arts. Pure mathematics is a very challenging field and abstract thinking is in itself an amazing form of expression. For centuries, most of the brilliant and complex demonstrations in this area were based on a few approximate plaster models, tentative at best in reflecting the complexity of the reasoning. Computer-aided visualization in scientific and mathematical visualization has brought a novel element in the concretization of ideas and theories. It has helped refine and clarify complex calculation. It has also provided researchers and audiences with a precise and accurate tool to understand both the outcome and the process leading to elaborate visual statements. Today, computer technology permeates all fields of research, whether in science, art, or visual communication. It complements the various technical requirements needed in each discipline and has also become a stepping stone in the creation of better and more effective communication.

The work I did on cubic surfaces was part of a larger ongoing project–the 12-30 project [11] focused on testing math-visualization programs in a graphic environment. In the course of the particular research for this project, I read a very instructive review [12] of a treatise written by A. Cayley on the four-color map problem. It was not unusual at that time for scientist to look into the physiological aspect of the art-making process as Newton, Fourrier and many others did. For Cayley, color was much more than a cosmetic device. It could be used for achieving clarity, making new discoveries and suggesting valuable ideas. It will be indeed a challenging direction for future work to revisit the Cayley cubic and other mathematical surfaces in various visual environments and according to the very color principles investigated by Cayley himself, back in the mid-1800s.

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