A Musical Scale Generated from the Ratio of Consecutive Primes

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Abstract

This paper describes a 12-note musical scale generated from the ratio of consecutive prime numbers that is used as the basis for the author's electronic composition *The Music of the Primes*. The scale is a type of extended just intonation, a tuning system whose intervals are derived from the harmonic (overtone) series and uses partials above the first six partials. The ratio of consecutive primes sequence makes a potentially infinite number of prime ratios available to the composer.

Introduction

The *Music of the Primes* is an electronic musical composition by the author [1] that was inspired by mathematician Marcus du Sautoy's book of the same title [2]. It is a computer-generated algorithmic composition encoded in Cycling '74's Max visual programming language [3]. All of the composition's musical parameters involve *prime numbers* [4]. The composer's main aesthetic goal was to saturate the pitch and rhythmic domains with perceivable prime number proportions that over the course of the work become gradually more complex as higher order primes are explored. In that sense, it is a musical exploration of the spacing of prime numbers. This paper focuses on a mathematical description of the musical scale on which the piece is based. The scale is related to the class of tuning systems called *extended just intonation* [5]. In such systems, all of the intervals are derived from the harmonic (overtone) series and thus may be exactly specified using whole-number ratios. The word extended refers to the use of partials above the first six partials. The music and intonations of twentieth-century American composers Harry Partch, Ben Johnson, La Monte Young, and James Tenney, among others, have served as inspirations for this work.

Prime Numbers

A positive integer p is *prime* if $p > 1$ and its only divisors are 1 and p. A positive integer greater than 1 that is not prime is said to be *composite*. The *prime sequence* begins

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots,$$

and we denote the nth prime number as p_n: i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, … [6]. The difference between two consecutive prime numbers g_n, called a *prime gap*, is defined as:

$$g_n = p_{n+1} - p_n.$$

The *prime gap sequence* is defined as:

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, \ldots.$$

We mention the prime gap sequence here because it plays a central role in the rhythmic structure of the work. The pitch domain, however, is exclusively derived from the ratio of consecutive primes.

For $n \geq 1$, we define the *ratio of consecutive primes* q_n as:

$$q_n = \frac{p_{n+1}}{p_n}.$$

We define the *ratio of consecutive primes sequence* as:

$$\frac{3}{2}, \frac{5}{3}, \frac{7}{5}, \frac{11}{7}, \frac{13}{11}, \frac{17}{13}, \frac{19}{17}, \frac{23}{19}, \frac{29}{23}, \frac{31}{29}, \frac{37}{31}, \frac{41}{37}, \frac{43}{41}, \ldots.$$

Just (5-limit) Intonation vs. Equal Temperament

A *just (5-limit) intonation* is defined as a tuning system whose pitches may be represented as $2^p \cdot 3^q \cdot 5^r$, where p, q and r are integers. For example, the 7 notes of a traditional diatonic scale on C are represented using 5-limit ratios in Figure 1.

Note	C	D	E	F	G	A	B	(C)
Ratio	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$

Figure 1: *A just (5-limit) tuning of a diatonic scale on C.*

This intonation utilizes "pure" ratios found in the lower part of the harmonic (overtone) series. A common way to extend this 5-limit system to include the 12 notes of a chromatic scale on C is shown in Figure 2 [7]:

Note	C	D♭	D	E♭	E	F	F♯	G	A♭	A	B♭	B	(C)
Ratio	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{15}{8}$	$\frac{2}{1}$
Cents	0	112	204	316	386	498	590	702	814	884	1018	1088	1200

Figure 2: *A just (5-limit) tuning of a chromatic scale on C.*

In general, if p is prime, a p-limit scale uses ratios involving numerators and denominators that are products of primes $\leq p$ [8]. The reader who requires a primer explaining how notes and intervals are traditionally represented in tuning theory is referred to Kyle Gann's article "Just Intonation Explained" [9].

The cent is a logarithmic unit of pitch interval. The size of an interval in cents (c) may be obtained from a ratio as follows [10]:

$$c = 1200 \log_2 \frac{f_2}{f_1}.$$

By definition there are 1200 cents in an octave. In *12-tone equal temperament* (12tet), a commonly accepted standard for tuning a modern piano, there are 100 cents in each semitone. Equivalently, 1 cent is 1/100 of a 12tet semitone. Cents make the comparison of interval sizes a simple matter. In this paper, all cent values are rounded to the nearest cent to simplify the comparisons. Readers who require cent values to 3 decimal places may use the formula above, or may consult Kyle Gann's lexicon of 700 pitches within the octave titled "Anatomy of an Octave" [11]. The size of the 12tet semitone is based on the irrational number $\sqrt[12]{2} = 2^{\frac{1}{12}} \approx 1.059$. For comparison with Figure 2, Figure 3 shows the 12 notes of a 12tet chromatic scale using cents and ratios expressed as simplified powers of 2.

Note	C	C♯/D♭	D	D♯/E♭	E	F	F♯/G♭	G	G♯/A♭	A	A♯/B♭	B	(C)
Ratio	$\frac{1}{1}$	$2^{\frac{1}{12}}$	$2^{\frac{1}{6}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{3}}$	$2^{\frac{5}{12}}$	$2^{\frac{1}{2}}$	$2^{\frac{7}{12}}$	$2^{\frac{2}{3}}$	$2^{\frac{3}{4}}$	$2^{\frac{5}{6}}$	$2^{\frac{11}{12}}$	$\frac{2}{1}$
Cents	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

Figure 3: *A modern piano tuning (12tet).*

The Ratio of Consecutive Primes Scale

To create an octave-based scale using the ratio of consecutive primes, we partition the *octave* (2/1) into discrete notes using q_n as a *generator*. This process is shown in Figure 4.

Note		q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	...	
Ratio	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{5}$	$\frac{11}{7}$	$\frac{13}{11}$	$\frac{17}{13}$	$\frac{19}{17}$	$\frac{23}{19}$	$\frac{29}{23}$	...	$\frac{2}{1}$

Figure 4: *Notes derived from the ratio of consecutive primes.*

As Euclid proved, an infinite number of primes exist. Therefore q_n can theoretically generate an infinite number of notes. Arranging the ratios in Figure 4 in ascending scalar order yields Figure 5.

Ratio	$\frac{1}{1}$	$\frac{19}{17}$	$\frac{13}{11}$	$\frac{23}{19}$	$\frac{29}{23}$	$\frac{17}{13}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{11}{7}$	$\frac{5}{3}$	$\frac{2}{1}$
Decimal	1.0	1.118	1.182	1.211	1.261	1.308	1.4	1.5	1.571	1.667	2.0
Cents	0	193	289	331	401	464	583	702	782	884	1200

Figure 5: *Notes q_1–q_9 arranged in ascending scalar order.*

Each note in Figure 5 is specified using three equivalent representations: (1) as a frequency ratio f_2/f_1 relative to the *reference pitch* 1/1; (2) as a decimal expansion rounded to three decimal places; and (3) using cents.

Figure 6 shows the tuning specification for *The Music of The Primes*.

Note	E♭	F	F♯	G♭	G	A♭	A	B♭	B	C	D♭	D	(E♭)
Ratio	$\frac{1}{1}$	$\frac{19}{17}$	$\frac{13}{11}$	$\frac{23}{19}$	$\frac{29}{23}$	$\frac{17}{13}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{11}{7}$	$\frac{5}{3}$	$\frac{23}{13}$	$\frac{13}{7}$	$\frac{2}{1}$
Cents	0	193	289	331	401	464	583	702	782	884	988	1072	1200

Figure 6: *The Music of the Primes tuning specification.*

All of the notes in this scale may be represented as

$$2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f \cdot 17^g \cdot 19^h \cdot 23^i \cdot 29^j,$$

where *a*–*j* are integers. Thus it meets our earlier definition of a just (*p*-limit) intonation with $p = 29$. The work's reference pitch E♭4 (C4 is middle C) is approximately 311.1 Hz. The tonic note E♭ is assigned to the reference pitch 1/1. Octave-related pitches are obtained by multiplying by 2^n, where n is an integer. Traditional musical letter names are strategically assigned to the other notes of the scale to balance both musical and mathematical concerns. Notice that Figure 6 adds two new ratios to Figure 5: 23/13 and 13/7. These ratios fill the 6/5 gap between 5/3 and 2/1. Higher order q_n (i.e., $n > 9$) were considered, but were ultimately rejected by the composer due to an aesthetic preference for the sonic signatures of 23/13 and 13/7, and a desire to stay within the *29-limit*.

To give the reader some impression of how the ratio of consecutive primes (ROCP) sequence is used to create melodies and harmonies, the first 2:25 of the work is briefly described as follows. Section A (0:55–1:37) introduces the work's main theme: E♭ (1/1), B♭ (3/2), C (5/3). Presented over a E♭–B♭ (3/2) harmonic drone, the main theme is repeated and then extended by a single note to include the note A (7/5). Section A concludes with the progression E♭–B♭ (3/2) moving to B♭–F♯ (52/33 = 13/11 · 2/1 · 2/3) and returning to E♭–B♭ (3/2) again. The intervals E♭–B♭ and B♭–F♯ may also be found in the work's introduction (0:00–0:55). After these two intervals are stated at the outset of the work, the B♭–F♯ (52/33) interval is transformed into B♭–G♭ (23/19), and then the introduction comes to rest on a tonic E♭ major harmony with a 29/23 major third. The main theme returns in the B section (1:37–2:25), this time in a new octave. The main theme is then extended to include the notes: B (11/7), F♯ (13/11), A♭ (17/13) and F (19/17). A summary of the melodic notes presented over the course of the A & B sections unveils the ROCP sequence segment: 3/2, 5/3, 7/5, 11/7, 13/11, 17/13, 19/17.

Concluding Remarks

The scale in Figure 6 features a number of interesting and rare intervals. Except for the relatively common just 3/2 perfect fifth, 5/3 major sixth, 7/5 septimal tritone, and 11/7 undecimal minor sixth (or augmented fifth), most of the intervals remain unnamed in Gann's lexicon [see 11]. Rare intervals include the hauntingly intoxicating 17/13 "perfect fourth", the 29/23 major third (which is exceptionally close to its 12tet counterpart), and exquisitely beautiful 19/17 whole tone. The ratios 13/11 and 23/19 form a wonderfully ambiguous interval pairing rife with musical implications. The non-adjacent prime ratios 23/13 and 13/7 complete the system and offer interesting possibilities for harmonization, seventh-degree resolution, and modulation that are all exploited in the *The Music of the Primes*.

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