Infinite Rhythmic Tiling Canons

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Abstract

Maximally-even (ME) rhythms, in which attacks (or the onsets of notes) are distributed as evenly as possible over a given number of pulses, are common in many cultures. One way to generate a ME rhythm with m attacks per n pulses is to digitize a line of slope m/n. If the slope is rational, the resulting rhythm is periodic; irrational slopes yield Sturmian sequences, which are balanced, almost periodic, self-similar and hierarchical sequences that are highly relevant in the study of musical objects possessing these same properties. Rhythmic tilings are combinations of rhythms that yield one and only one onset per pulse. Rhythmic tiling canons are tilings in which the component rhythms are version of a single tile. Sturmian rhythms can be used to create completely new kinds of aperiodic rhythmic tiling canons, in which the relations of the component rhythms are determined by the continued fraction of the slope.

Sturmian Rhythms

Maximally-even (ME) rhythms [6], in which attacks are distributed as evenly as possible over a given number of pulses, are common in many cultures [8]. One way to generate a rhythm with a ME distribution of m attacks over n pulses is to take the *lower mechanical sequence* of slope $a = 1 - \frac{m}{n}$, a discretization of the line $y = ax$ [9]. Given $a \in \mathbb{R}$, the lower mechanical sequence of slope a, $c_a : \mathbb{N} \to \{0, 1\}$, is given by

$$c_a(n) = \lfloor (n+1)a \rfloor - \lfloor na \rfloor - \lfloor a \rfloor \ .$$

If the slope is rational, this sequence will yield a periodic ME rhythm. For instance, if $a = 5/9$, then the sequence will be a repetition of 01 01 01 011. Associating attacks with 0s in the sequence yields the $2 + 2 + 2 + 3$ rhythm of Dave Brubeck's *Blue Rondo à la Turk*, which is a ME distribution of four attacks over nine pulses.

Irrational slopes, where $a \in \mathbb{R} \setminus \mathbb{Q}$, yield *Sturmian sequences*, which are balanced, almost periodic, self-similar and hierarchical sequences that are highly relevant in the study of musical objects possessing these same properties [5]. For example, the Sturmian sequence of slope ϕ is $c_\phi = 0101101011011\ldots$, which can be separated into groups beginning with each 0. These groups are of length two or three (Figure 1).

These groups can be separated into hierarchical levels containing short and long groups as shown in Figure 2. Note the while the durations of level one are equal, at higher levels the ratio of long to short durations converges on ϕ. The sequence of events at each level is an augmentation of the sequence of the previous level. This is possible because of the periodicity in the continued fraction for ϕ [2, 3, 4].

Rhythmic Tiling Canons

A rhythmic tiling is a combination of rhythms in which one and only one onset occurs per pulse [7]. A rhythmic tiling canon is a tiling in which the component rhythms are version of the same rhythm, transformed

c_a = 0101101011011010110101101101011011

c_a = 01 011 01 011 011 01 011 01 011 011 01 011 011

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c_a = [(01 011) (01 011 011)] [(01 011) (01 011 011) (01 011 011

Figure 1: *Digitization of a line of slope $a = \phi$. The mechanical sequence digitizes the line as a sequence of 0s and 1s. Associating "lower" attacks with 0s and "higher" attacks with 1s yields the musical sequence at the bottom of the figure.*

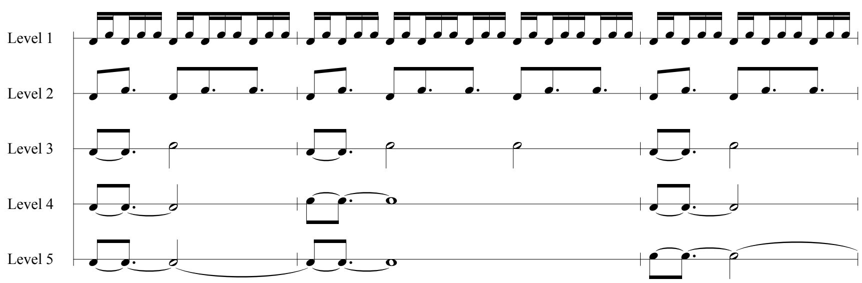
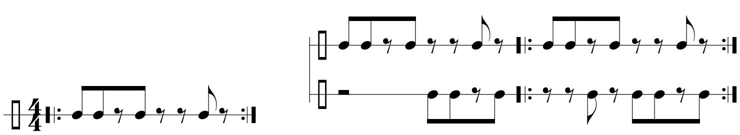


Figure 2: *Hierarchical levels of c_ϕ. Associating "lower" attacks with shorter groupings yields identical sequences (disregarding duration).*

by simple delay, scaling (augmentation or diminution), and/or retrograde [1]. In Figure 3 a simple rhythm is combined with a delayed version to create a common rhythmic tiling canon.[1]



(a) (b)

Figure 3: *The rhythmic pattern or tile in (a) combines with a delayed version of the tile to form a rhythmic tiling canon in (b).*

Sturmian rhythms can be used to create completely new kinds of aperiodic rhythmic tilings, in which the canonic structure of the component rhythms is determined by the continued fraction of the slope [2]. The typical rhythmic tiling is periodic with tiles of finite length. For tilings based on Sturmian rhythms both tiles and tiling are aperiodic and the composite rhythm is often a diminution of one or more tiles.

Figure 4 shows the rhythmic structure for a tiling canon by Tom Johnson in which successive augmentations are delayed so as to fill in gaps left by previous voices. This tiling pattern can be adapted to Sturmian canons as demonstrated in Figure 5. Successive voices are successive levels of the sequence $c_{\sqrt{2}}$ with onsets for the shorter durations and rests for the longer durations. As with Johnson's canon, successive voices are delayed so as to fill gaps left by previous voices. Gaps in Figure 5 would be covered by higher-level voices.

Figure 6 shows an original infinite tiling canon based on the Sturmian sequence of slope ϕ. Successive voices are progressive near augmentations of previous voices. Some attacks are omitted in the canon line for musical purposes. Nonetheless, by the end of the excerpt the eighth-note pulse is fully tiled by the

[1]This pattern is the drum rudiment known as the "paradiddle" and occurs throughout much of Steve Reich's *Different Trains*.

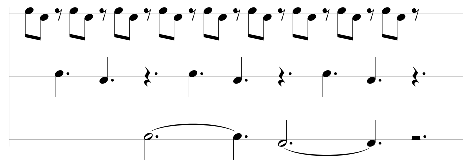


Figure 4 : *Rhythmic structure for tiling canon from on of Tom Johnson's Rational Melodies. Lower voices are progressive augmentations by a factor of three and are delayed so as to fill the gaps left by higher voices. The unison rests at the end of the excerpt would be the beginning of the fourth voice.*

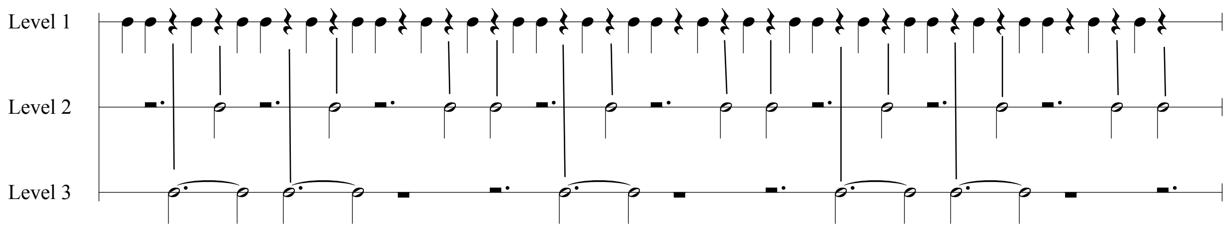


Figure 5 : *Tiling pattern from Figure 5 adapted for* $c_{\sqrt{2}}$.

combination of the canon line proceeding at four different speeds.

References

[1] E. Amiot, "Structures, algorithms, and algebraic tools for rhythmic canons", *Perspectives of New Music*, 49 (2): 93–142, 2011.

[2] C. Callender, "Sturmian Canons", In *Mathematics and Computation in Music*, Lecture Notes in Computer Science, Vol. 7937: 6475, 2013.

[3] D. Canright, "Fibonacci Gamelan Rhythms", *1/1: The Journal of the Just Intonation Network*, 6 (4), 1990.

[4] N. Carey and D. Clampitt., "Self-similar pitch structures, their duals, and rhythmic analogues", *Perspectives of New Music* 34 (2): 62–87, 1996.

[5] D. Clampitt, M. Domínguez, and T. Noll,. "Plain and Twisted Adjoints of Well-Formed Words", In E. Chew, A. Childs, and C.H. Chuan (eds.) *Proceedings of the Second International Conference on Mathematics and Computation in Music*, pp. 65–80. Springer, Berlin, 2009.

[6] J. Clough and J. Douthett, "Maximally Even Sets", *Journal of Music Theory* 35: 93–173, 1991.

[7] J. Mingo, "A classification of one dimensional almost periodic tilings arising from the projection method." *Transactions of the American Mathematical Society*, 352 (11): 5263–5277, 2000.

[8] G. Toussaint, *The Geometry of Musical Rhythm: What Makes a 'Good' Rhythm Good?*, Chapman and Hall, 2013.

[9] H. Wehlou, "Run-hierarchical structure of digital lines with irrational slopes in terms of continued fraction and the Gauss map", *Pattern Recognition* 42: 2247–2254, 2009.

Figure 6 : *Original tiling canon based on a Sturmian sequence of slope* ϕ*. By the time the fourth voice enters, the composite rhythm of all four voices yields a sequence of continuous eighth notes.*