Unexpected Beauty Hidden in Radin-Conway's Pinwheel Tiling

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Abstract

In 1994, John Conway and Charles Radin created a non-periodic ***Pinwheel Tiling*** of the plane using only 1 by 2 right triangles. By selectively painting either every fifth triangle or two out of every five triangles, based only upon their location in the next larger triangle, one can discern 15 unexpected and distinctive patterns. Each of these patterns retains the non-periodic nature of the original tiling.

Introduction

In 1994, John Conway and Charles Radin created a tiling of the plane using only 1 by 2 right triangles. Mathematically, this ***Pinwheel Pattern*** is very interesting because it produces a non-periodic tiling by using tiles that are all identical. However, as we can see in the bottom triangle in Figure 2, the resulting tiling is too uniform to be artistically interesting. In this paper we seek to uncover patterns and beauty where neither is apparent.

The pinwheel tiling is constructed by recursively subdividing 1 by 2 right triangles into five smaller triangles by following the guidelines shown in Figure 1. Since each new triangle is similar to the original triangle, the process can be repeated indefinitely. This figure shows the first, second, and fourth iteration. The fourth iteration is shown twice, first highlighting all previous subdivisions, and then without highlighting any previous subdivisions.

One method for seeking a pattern is to use an idea applied to Penrose Tilings. We draw a quarter arc centered at the right-angled vertex with radius half the width of each tile. Figure 2 shows a single such tile, and then the first, second and fourth iteration of this process. The fourth iteration is shown twice, first showing all $5^4 = 625$ triangles, and then showing only the arcs.

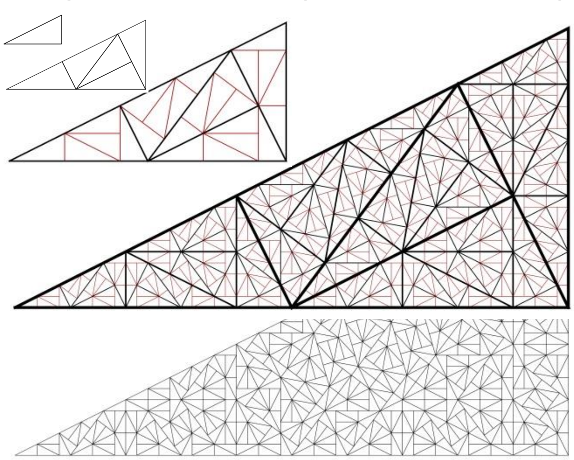

Figure 1: *Pinwheel Tiling Subdivisions*
Iterations $0, 1, 2, 4$ *and* $4'$

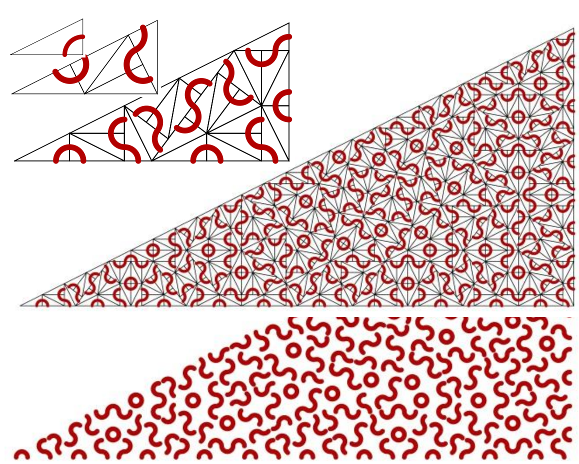

Figure 2: *Quarter Arcs*
Iterations $0, 1, 2, 4$ *and* $4'$

We now explore a totally different method for uncovering patterns. At the final iteration, we label each triangle A, B, …, E, based upon its location in the subdivision of the next larger triangle using the labeling shown in Figure 3. By simply painting the interior of only one type of triangle we obtain the unexpected and distinctive patterns shown in Figures 4, 5, and 6. By painting the interior of two types of triangles, Figure 4 and Figure 5 merge to become the pattern shown in Figure 7. And Figure 4 and Figure 6 merge to become the pattern shown in Figure 8.

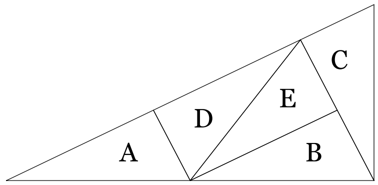

Figure 3: *Pinwheel Tiling Subdivision Labeling*

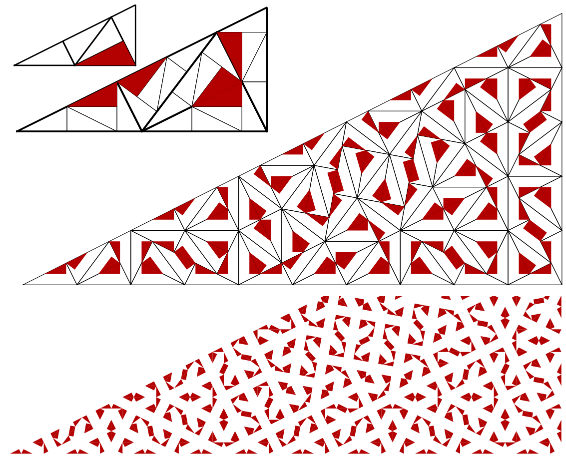

Figure 4: *Painting all **B** triangles Iterations* 1, 2, 4 *and* 5

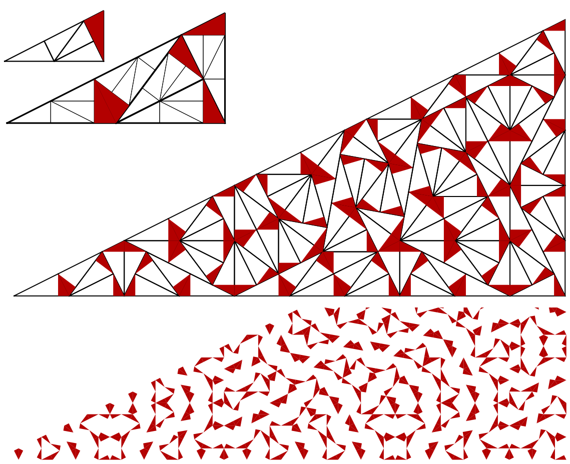

Figure 5: *Painting all **C** triangles Iterations* 1, 2, 4 *and* 5

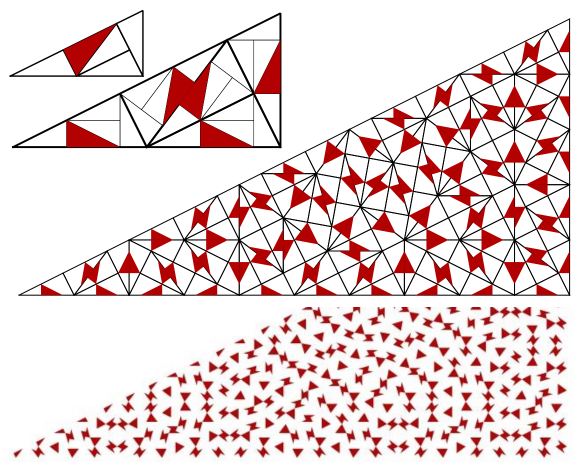

Figure 6: *Painting all **D** triangles Iterations* 1, 2, 4 *and* 5

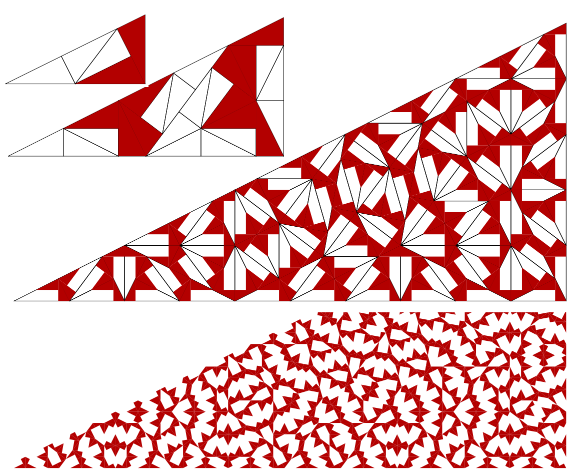

Figure 7: *Painting all **B** and all **C** triangles Iterations* 1, 2, 4 *and* 5

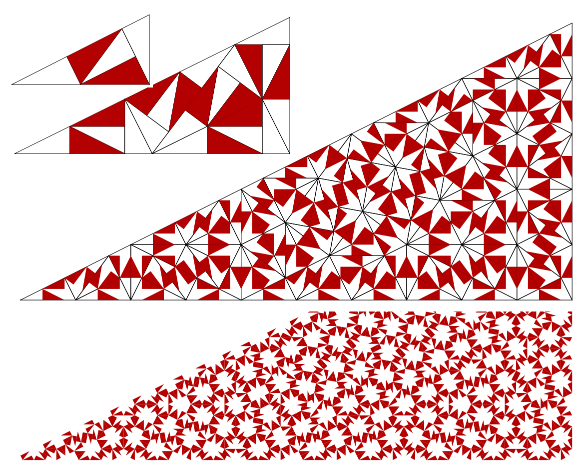

Figure 8: *Painting all **B** and all **D** triangles Iterations* 1, 2, 4 *and* 5

We now create two pieces of art using the ideas discussed above. Figure 9 shows a piece titled ***A Quilted Triangle***. The large triangle is first cut into $5^7 = 78{,}125$ triangles by iterating the Pinwheel Tiling 7 times. The edges of all of the $5^6 = 15{,}625$ triangles in the sixth iteration are shown. Along both legs of the large triangle and especially visible at the three vertices, the tiny triangles in the 7^{th} iteration are drawn. Next, we paint about 30% of the 15,625 triangles by dividing the large triangle into five rows and five columns, ignoring regions at each end. In the left column, we paint the interior of every triangle A. In the second column, we paint the interior of every triangle C. We continue with E and then B and finally D. We use the same sequence, A, C, E, B, and D for painting the interiors of the triangles in the rows starting from the bottom row. Thus, both the B triangles and the D triangles are painted in the intersection of the fourth row from the bottom and the fifth column from the left. This rectangular portion of the triangle is the daisy pattern shown in Figure 8 above. In addition to each pattern shown on the previous page, this triangle contains two more single and eight more double patterns. Thus, it contains all five single triangle type patterns and all ten double triangle type patterns. Note that highlighting three triangle types simply produces the negative of the double triangle types. That is, the pattern created by painting the interior of the A, the C, and the E triangles would produce the negative image of the pattern created by painting the B and the D triangles.

The artwork also contains some of the recursive process of the Pinwheel Tiling. The edges of all five triangles in the first iteration are drawn with thick black lines, with the exception that the hypotenuse of A and the short edge of B and C type triangles are thick yellow lines. In each of the following iterations, only the edges of the five triangles in the central triangle are emphasized. The yellow lines produce a spiral or *pinwheel* pattern. The third column was specifically chosen as type E so that the innermost triangle is painted. Note that the bold lines never cut through a triangle. By the iterative process, any triangle which is adjacent to a bold line must have one of its three edges lying on the bold line.

In the darker rectangular regions, two out of every five triangles are painted. In the lighter rectangular regions around the edge, and in the triangular regions along the hypotenuse, exactly one out of every five triangles is painted.

In Figure 10, we start by subdividing a large triangle using nine iterations to get almost two million triangles. We then trim a square carpet with fringe out of the interior of the triangle. Similar to Figure 9, the central region of the carpet has five rows and five columns, one for each type of triangle. Although hard to see in a black and white printing, the original artwork includes narrow rows and columns of red to add to the aesthetic appeal.

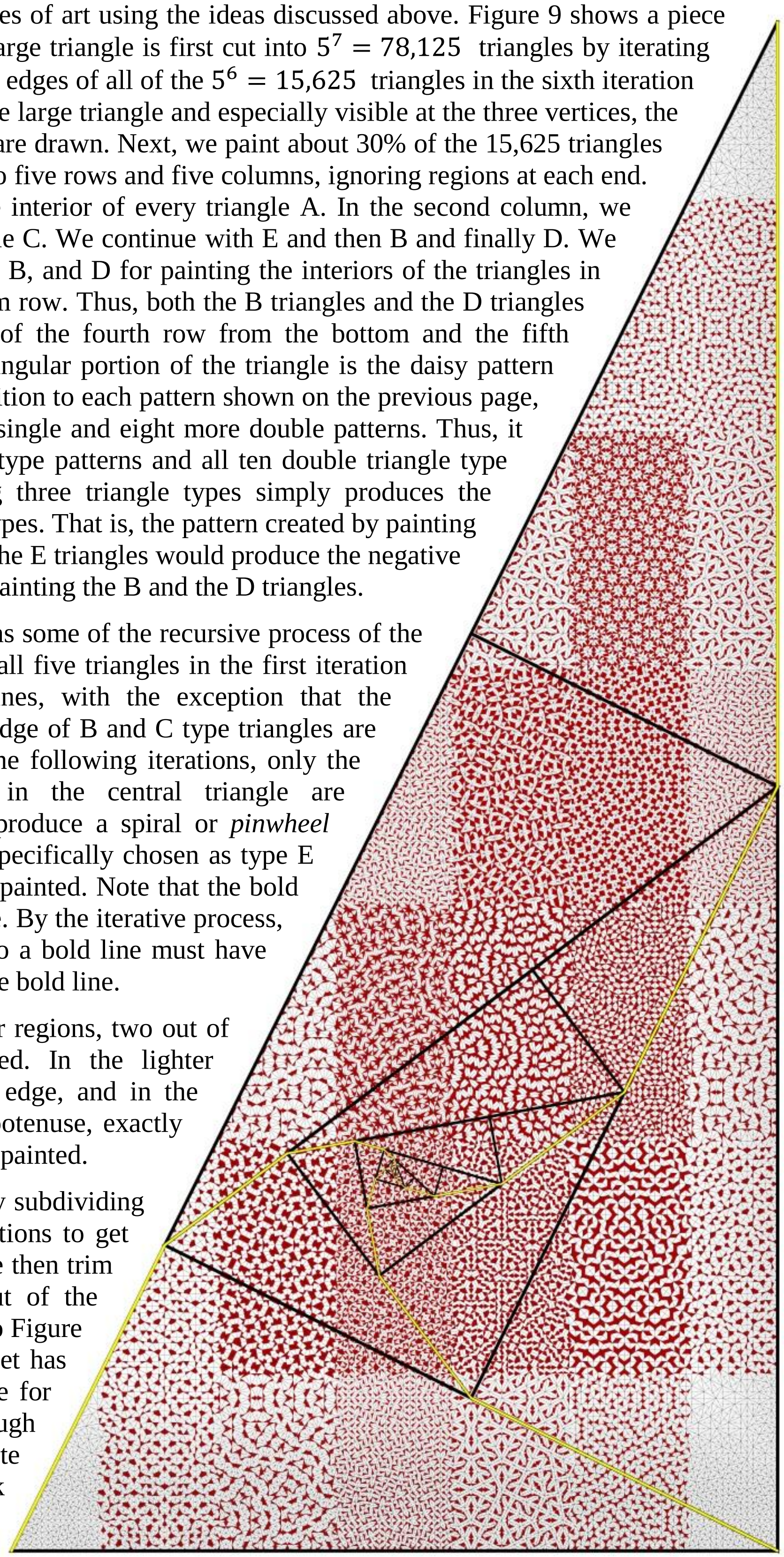

Figure 9: *A Quilted Triangle*

Notice that this trimming process does not cut any triangles. Instead of cutting after painting, we paint only the triangles whose center point is in the desired region. Since no triangles are ever moved or rotated, this carpet can be placed back into the Pinwheel Tiling and it would fit precisely.

While the Pinwheel Tiling is mathematically interesting, its apparent lack of artistic interest is deceiving. Moreover, each of the 15 patterns created by selectively painting all triangles of a given type or a given pair of types is very distinctive and easily identifiable. Interesting patterns within the tiling become visible and the non-periodic nature of the tiling begins to become apparent.

References

Charles Radin, *The Pinwheel Tilings of the Plane*, Annals of Mathematics, Vol. 139, 1994, pp. 661-702.

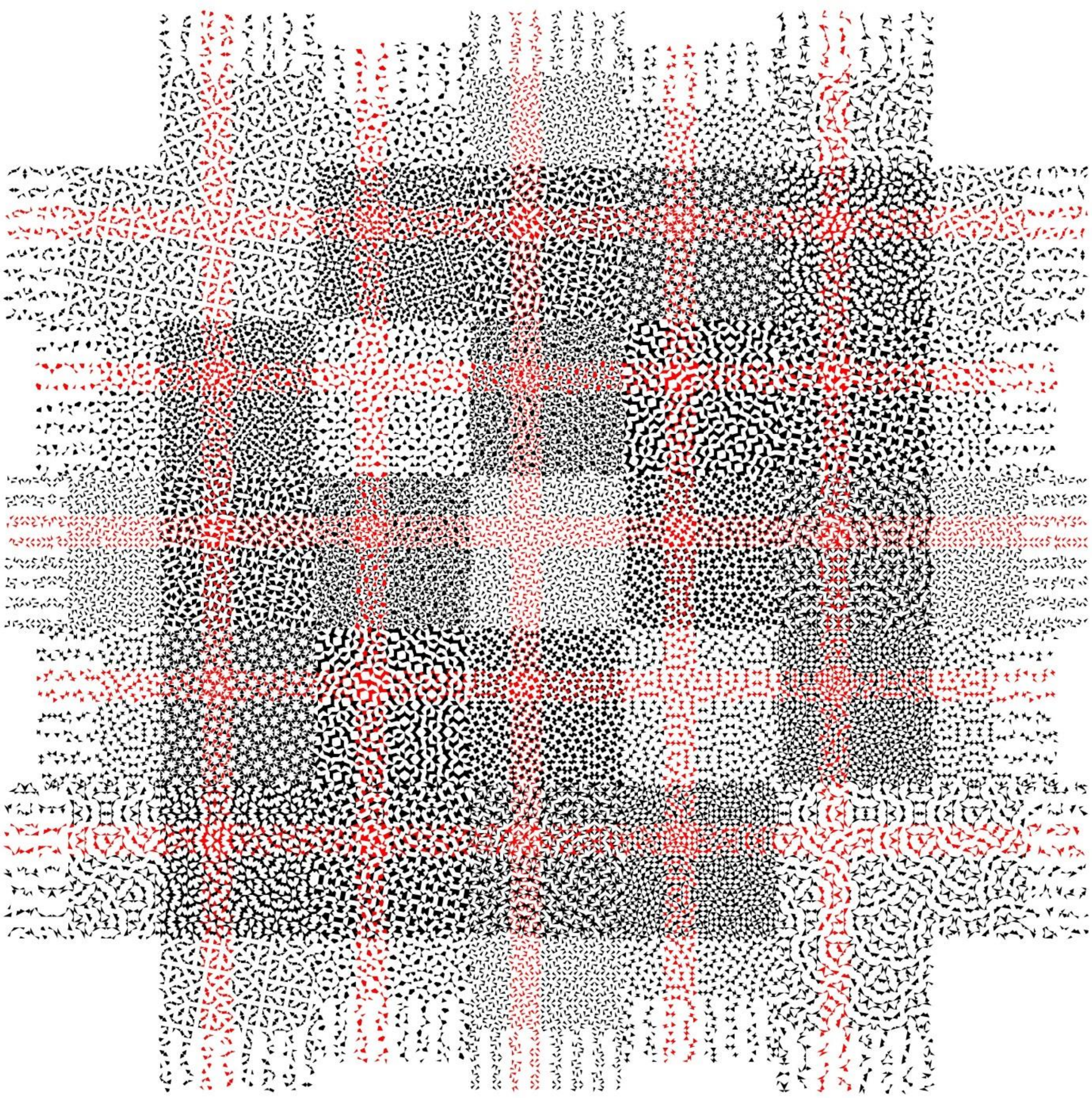

Figure 10: *A Radin-Conway Quilt*