Surfaces with Natural Ridges

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Abstract

We discuss surfaces with singularities, both in mathematics and in the real world. For many types of mathematical surface, singularities are natural and can be regarded as part of the surface. The most emblematic example is that of surfaces of constant negative Gauss curvature, all of which necessarily have singularities. We describe a method for producing constant negative curvature surfaces with prescribed cusp lines. In particular, given a generic space curve, there is a *unique* surface of constant curvature $K = -1$ that contains this curve as a cuspidal edge. This is an effective means to easily generate many new and beautiful examples of surfaces with constant negative curvature.

Mathematical surfaces with singularities. The basic mathematical model for a surface is a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies a *regularity condition*, namely that the partial derivatives f_x and f_y are linearly independent vectors at every point. These vectors constitute a basis for the tangent space at each point, and, to first order, the surface looks like a plane. The regularity condition is equivalent to requiring that the vector field $f_x \times f_y$ is never zero, and the *normal N* is the unit vector in this direction. In many situations it makes sense to allow this cross-product to become zero at some points, and instead ask only that there be a well-defined normal N, i.e., a unit vector field that is orthogonal to f_x and f_y at every point. Such a generalized surface is called a *frontal*, and the points where $f_x \times f_y = 0$ are called *singularities*.
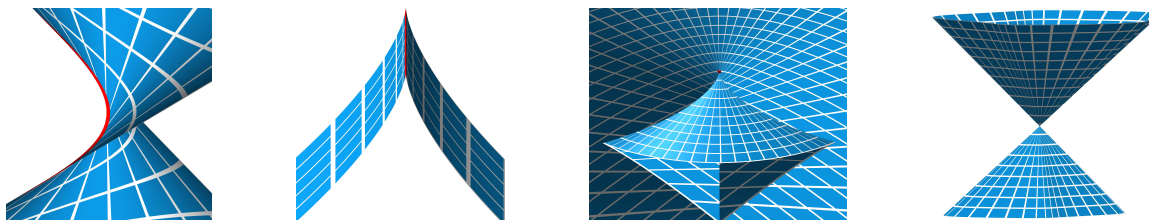

Figure 1 : *(a) Tangent developable, (b) cuspidal edge profile, (c) swallowtail and (d) cone singularity.*

An instructive example is that of a *tangent developable*. Take a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, with non-vanishing curvature and torsion, e.g., a helix. The map $f(x, y) = \gamma(x) + y\, \gamma'(x)$ is a *zero-curvature* ruled surface, wherever it is regular. The cross-product $f_x \times f_y = y\, \gamma''(x) \times \gamma'(x)$ vanishes precisely along $C = \{y = 0\}$, the singular curve. The image $f(C)$ is just the curve γ. This singularity is called a *cuspidal edge* (Figure 1 (a), (b)). The smooth vector field $N = (\gamma' \times \gamma'')/\|\gamma' \times \gamma''\|$, actually the unit binormal of the curve, is orthogonal to both f_x and f_y, and so the surface is a frontal. This example is important because all zero-curvature surfaces other than cones and cylinders are locally given as tangent developables.

Another common example is a *swallowtail* singularity (Figure 1(c)), where the image of the singular curve itself has a cusp point on it. A less common, but easy to understand singularity is a *cone point* (Figure 1(d)). This can be constructed by taking a plane curve $\gamma(t) = (a(t), b(t), 1)$ and then adding all the lines through $(0, 0, 0)$ and points on the curve, by the formula $f(x, y) = y\, (a(x), b(x), 1)$. The entire line $y = 0$ in the parameter space maps to a single point $(0, 0, 0)$. The normal is well defined by $N = (\gamma' \times \gamma)/\|\gamma' \times \gamma\|$.

Ishikawa and Machida [5] showed that the *stable* singularities for constant curvature surfaces are cuspidal edges and swallowtails. Stable, or *generic*, means that if you deform the surface slightly, keeping the same constant curvature, the new surface will also have the same type of singularity.

Physical surfaces with singularities. You probably won't see many swallowtails in the real world. But cone points appear in many places, and approximate cuspidal edges can arise when a series of surfaces are connected together (Figure 2, far left). Mathematical singularities can also express themselves indirectly: for example, Ben Amar et al. [1] study a physical model of the growth of flower petals. Where the mathematical model for the surface has natural singularities, they find in one case that structural deformation occurs, manifesting itself in edge curling of petals with negative curvature. In the case of positive curvature, they find that singularities of the mathematical surface correspond to strong veins in the petals.

Figure 2 : *Approximate singularities in nature.*

One way to *construct* physical cuspidal edges is illustrated by an open paperback book, which is essentially a zero-curvature surface with a cuspidal edge running down the spine.

In Western architecture you are more likely to see squared corners than the sharp ridge of a cuspidal edge. But in East Asia the roofs of temples and palaces (Figure 3) usually curve up sharply at the top and along corners to create the effect of a stylized cuspidal edge, often emphasized with decorative features.

Figure 3 : *Roofs at temple complexes in China (left) and Japan (right).*

Negatively curved surfaces. At a given point on a surface, there is a unique curve in any tangent direction that bends only as much as it is forced to by the surface itself. The curvature of this *geodesic* captures the curvature of the *surface* in that direction. At a typical point, the maximum and minimum such curvature values, κ_1 and κ_2, are the *principal curvatures*. The *Gaussian curvature* of the surface is the product $K = \kappa_1 \kappa_2$. If this is negative, then κ_1 and κ_2 have opposite sign and the principal curves are bending in opposite ways away from the tangent plane, so the surface has a saddle shape (Figure 4).

If you imagine a surface that has negative curvature *everywhere*, with these (necessarily orthogonal) curves bending away from each other at every point, you might feel that you will run into trouble after a while. And you would be right: a surface is called *complete* if you can follow a geodesic in any direction forever. A strong theorem of Efimov [3] states that there does not exist a *complete*, regular, surface in $\mathbb{R}^3$ of Gaussian curvature $K(x, y) < -c^2 < 0$, where c is any non-zero number— even if we allow self-intersections! We might be able to follow some geodesics forever in *some* direction, but what happens in general is that we come to a singularity of some kind, like the cuspidal edges and cone points in Figure 4.

Pseudospherical surfaces. A surface with constant Gaussian curvature $K = -1$ is called a *pseudospherical* surface. (The theory is the same for any choice of negative constant K). After hyperbolic geometry was
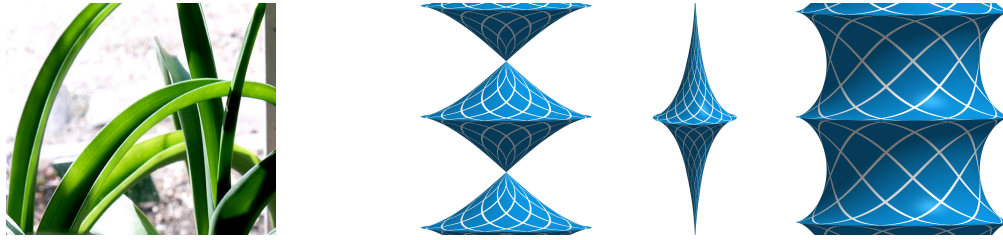
Figure 4 : *A plant with negatively curved leaves, and the three types of constant negative curvature surfaces of revolution, with asymptotic curves shown. The middle one is the* pseudosphere.

discovered in the 19th Century, a natural question arose whether is was possible to realize the complete hyperbolic plane as a geometric submanifold of Euclidean 3-space. Hilbert [4] proved in 1901 that this is impossible, and this was later generalized by Efimov to the result mentioned above.

It was also known since the 19th Century that pseudospherical surfaces are naturally associated with the *sine-Gordon equation* $\frac{\partial^2}{\partial x \partial y}\phi = \sin\phi$, where ϕ is the angle between certain *asymptotic* coordinate lines. An asymptotic curve is one which has no curvature in the direction normal to the surface. Solutions of the sine-Gordon equation correspond to pseudospherical "surfaces", but along the curves where $\phi = n\pi$, for integers n, the angle between the coordinates is zero and the surface is not regular. Methods are available for solving the sine-Gordon equation, and it is natural to use them, and to call the associated "surfaces" generalized pseudospherical surfaces. These are frontals, and this is one of the ways that frontals arise naturally from a mathematical point of view.
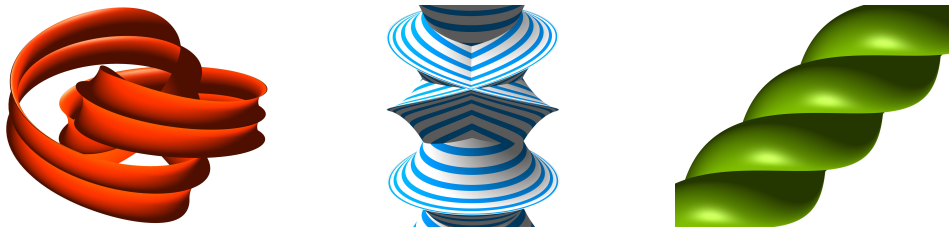

Figure 5 : *Pseudospherical surfaces containing closed curves and (right) helices as cuspidal edges.*

Generating pseudospherical surfaces from space curves. An interesting phenomenon appears when we include singularities within pseudospherical surfaces. Just as with zero-curvature surfaces, which are mostly locally represented by tangent developables, we can use space curves to *generate* pseudospherical surfaces. Given a space curve with non-vanishing curvature $\kappa \neq 0$, and torsion τ that satisfies $|\tau| \neq 1$ (compare with $\tau \neq 0$ for *zero*-curvature surfaces), there is a *unique* pseudospherical surface that contains this curve as a cuspidal edge! This is pointed out by Popov [6]. Lack of space restricts us to saying here only that it follows from uniqueness of solutions to a Cauchy problem for the sine-Gordon equation, but see [6, 2].

Recently one of the authors investigated the general problem of constructing pseudospherical frontals with prescribed singularities [2]. Put briefly, methods are found to construct all stable singularities, all singularities where the singular set is a regular curve in the coordinate domain, and various "degenerate" singularities, which exhibit features such as two singular curves crossing (Figure 7). For example, the statement above extends to curves where $\kappa = 0$ or $|\tau| = 1$ at a point. The solution still exists, but the singular curve is degenerate at such a point (Figure 7). The solutions can be computed numerically, using methods that grew out of soliton theory.[1] Some examples constructed using this technique are shown in Figures 5, 6 and 7. The curves shown in Figure 6 are asymptotic coordinate lines. In Figure 7, the last surface is generated

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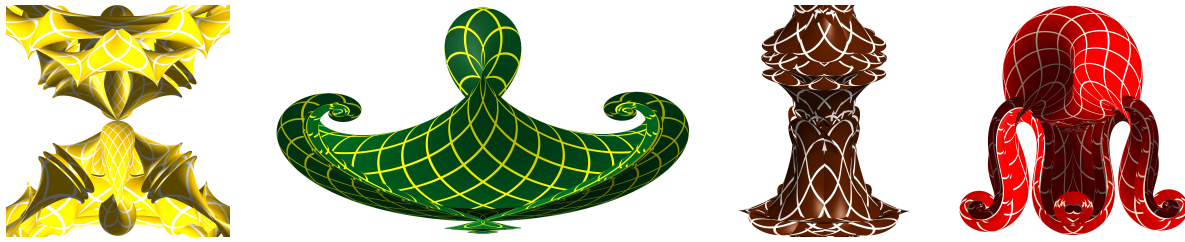[1]Code, *ksurf*, currently available at http://davidbrander.org/software.html

Figure 6 : *A cone singularity and the three pseudospherical surfaces generated by the planar curves with, in order, $\kappa(s) = \exp(s^2)$, $\kappa(s) = 1.6 + \cos(s)$ and $\kappa(s) = 2 - s^2$.*

by the Viviani curve $\gamma(t) = 0.1(1 + \cos(t), \sin(t), 2\sin(t/2))$, which has torsion $\tau(t) = \pm 1$ at four points; singularities on this surface are highlighted by a truncated mean-curvature colouring.

We have computed many more examples, and clearly this singular curve method is a fruitful way to generate in a controlled manner many interesting new examples of pseudospherical surfaces. More generally, we argue that the presence of singularities is obviously natural in many situations, and the sharp definition added gives the surfaces an interesting structure and appearance.
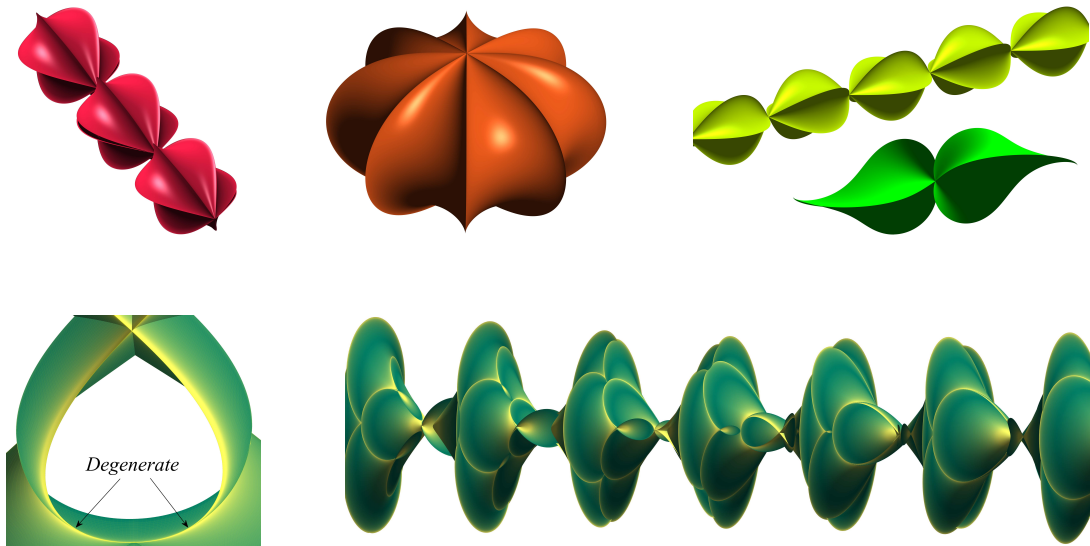

Figure 7 : *Top: surfaces generated by plane curves with $\kappa(s) = \cos^3(s)$ (left), $\kappa(s) = \cos(as)$ for special values of a (middle two), and $\kappa(s) = s$. Bottom: Viviani figure-8 space curve.*

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