The "ΦTOP": A Golden Ellipsoid

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Abstract

The golden ratio can be incorporated into an ellipse. Rotated around its major axis, the resulting 3-dimensional figure can be called a "golden ellipsoid". This object can act as a new kind of spinning top I call the "ΦTOP". The golden ellipsoid can be made into elegant sculptures as well. It also demonstrates an unusual visual illusion.

Introduction

Spinning tops and geometrical objects have each been of interest to many cultures spanning at least the past two millennia. (For a review of spinning tops, as well as to the literature of the subject, see [1]). Occasionally the Platonic solids, cones and some other regular three-dimensional geometrical shapes have been incorporated into spinning tops. Here we report the outcomes of a project exploring the physics, mathematics, art, aesthetics and psychophysics of spinning tops that utilize a special prolate ellipsoid.

The Tippe-Top, Kelvin's Pebbles, Jellett's Eggs and Shiva's Lingam Stones

Study of the problem of the rise of the center of mass (COM) of spinning objects is usually said to have begun in the late nineteenth century. These early mathematical treatments aimed to explain the motion of the newly invented and patented "tippe top" (Fig 1a). This semi-spheroidal top will invert when spun on a smooth surface while raising its COM. Because of the importance of friction in its dynamics, such a nonholonomic system is not readily amenable to analytic treatment, or of intuitive understanding. In notes written in 1844 — well before the invention of the tippe top in 1891 — William Thomson (later, Lord Kelvin) appears to have been the first person to analyze the problem of the rising COM of spinning objects. As a young student, he experimented with both oblate and prolate ellipsoidal pebbles (Fig. 1b), but never did a complete analytic theoretical treatment of the problem. J. H. Jellett, in his 1872 book *Theory of Friction*, provided a partial account of the related problem of the rise of the COM for an egg-shaped (ovoid) object (Fig.1c), making use of a new (adiabatic) invariant of the motion that he devised.

Figure 1 (from left to right): *(a) aluminum tippe top; (b) natural pebble; (c) polished agate stone egg; (d) classic Lingam stone; (e) ruby/fuschite Lingam stone; (f) black Lingam stone. Each is about 5 cm tall.*

Naturally occurring prolate ellipsoidal “Lingam stones” (Fig. 1d) from the Narmada River in India exhibit similar counter-intuitive dynamical behavior. When spun around their minor axis on a smooth horizontal plane, some Lingam stones (Fig. 1d and 1e) can stand erect and spin around their major axis in a vertical position. Most of them (such as Fig. 1f), however, will not stand upright. Is there an optimal prolate spheroidal shape that will stand erect if spun? After experimenting with natural pebbles, polished stones and rapid prototyped prolate ellipsoids, I found a range of optimal parameters for the prolate ellipsoids.

Ellipsoids, Ovoids and Related Objects

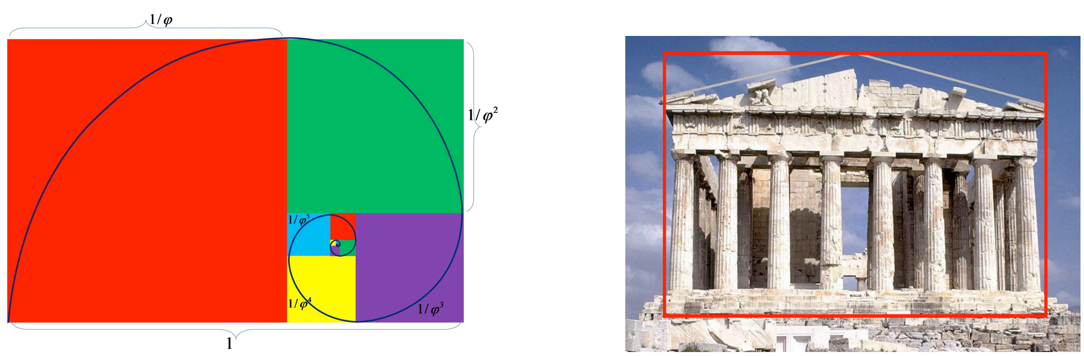
The equation for an ellipse is given by $x^2/a^2 + y^2/b^2 = 1$. A spheroid or ellipsoid has a surface defined by:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

where a, b and c are constants. Taking a = b, one has an oblate or prolate ellipsoid. Experiments done for this project have found that prolate ellipsoids with ratio c/a in a range from ~ 1.5 - 1.7 optimize the novel dynamics discussed in the previous section. (Note: Generalizing the equation of an ellipse to allow the powers of the co-ordinates to be different from 2 leads to a Lame curve, which can be also extended to three dimensions. Ovoids, or egg shaped objects, do not have such a well-defined mathematical basis.)

Mathematics and the Golden Mean

The “golden mean” — also called the “golden ratio” or the “golden section” — is designated by the Greek letter ϕ that is equal to about 0.61803398… Study of its many properties dates back at least 2500 years (cf. [2] for a good discussion of the subject). It can most easily be derived from the relation $1/\phi = 1 + \phi$. The “golden rectangle” (Fig. 2a) has the ratio of its side lengths equal to this number.



Figures 2a (l.), and b (r.): *(a) geometry of the golden section; (b) Parthenon with golden rectangle.*

Golden Mean, Art and Aesthetics

It has been claimed that the golden mean was included in a variety of art works and architectural monuments dating back to the Greek temples such as the Parthenon (Fig. 2b) — which ϕ does not fit well — and in the paintings of Leonardo da Vinci. Some cases that fit may have been conscious choices by the artists or architects; others may just be mathematical coincidences. If ϕ was consciously included, why? The 19th century physicist Gustav Fechner conducted the first quantitative study of an aesthetic preference for the golden mean by employing a set of rectangles. In the intervening years, many further studies have questioned his conclusion that there is a preference for the golden rectangle. As part of this project, I have conducted an informal preference survey using ellipsoids with c/a from 1.0 - 2.2. I found that the most selected c/a ratios were from 1.5 - 1.7, including what can be called the “golden ellipsoid”.

Introducing The "ΦTOP": A Golden Ellipsoid

Taking the equation for an ellipsoid and setting the ratio c/a equal to Φ which is the *inverse* of the often defined "golden mean" (or "golden ratio" or "phi" or "ϕ" - lower case), $c/a = \Phi = 1/\phi = (1 + 5^{1/2})/2 \sim 1.61803398$…..) one has what can be called the "golden ellipsoid" - a uniquely shaped solid body. This object exhibits near optimal rotational performance in the physical sense discussed above. It explicitly includes the golden mean in its shape, and looks visually appealing.

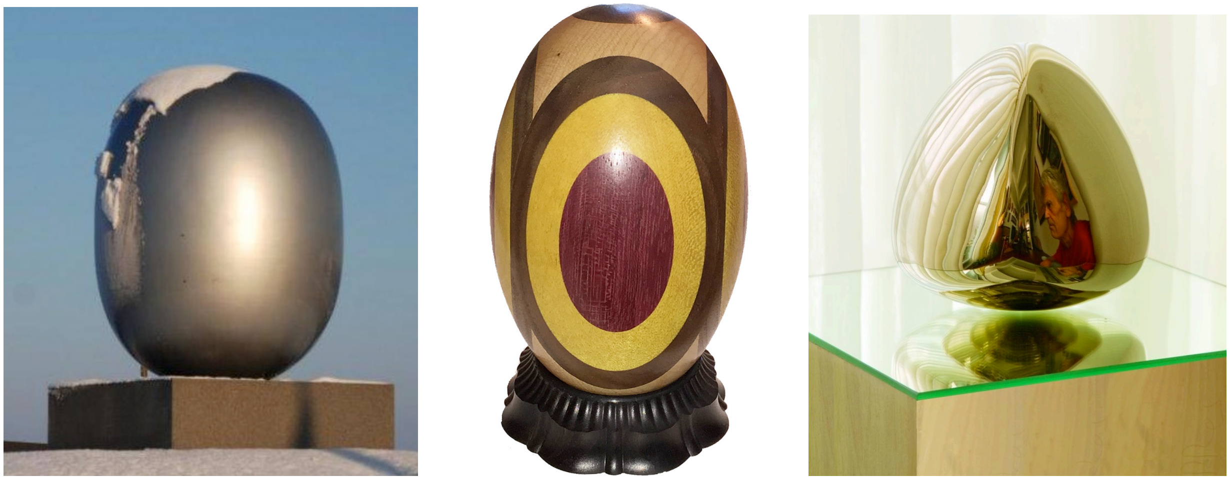
Figure 3. *A brass version of the ΦTOP with major axis ~ 5.1 cm, weighing ~ 220 grams.*

I have now designed and had fabricated versions of this unique spinning top and have named it the "ΦTOP". A machined brass version is shown above. In settling on a final design for the top, there were several human and mechanical design considerations. First, it should be of a size such that it could be easily spun using a person's fingers. If it has a length of about 5 cm, by placing a thumb of one hand on one side of the long axis and placing the index finger of the other hand on the opposite side, the top can easily be spun rapidly in a horizontal plane. If the object is much smaller or larger, it is more difficult to spin up. Second, the object should be made of a material of suitable mass, density and frictional coefficient so that it can actually stand up before friction damps the rotation. Metals such as brass, aluminum, stainless steel and copper have been tried, as well as plastics such as Delrin, acrylic and PVC. All work quite well, though the aluminum ΦTOP is the most easily spun up and spins the longest. A few seconds after being spun, the ΦTOP will stand upright. Once standing erect, it spins about twice as fast as when it was spinning horizontally (owing mainly to the difference of the moments of inertia about the major and minor axes). After ~ 60 - 120 seconds, it begins to settle down again owing to friction.

Some further considerations. The ΦTOP has a shape fairly close to that of a standard European rugby ball (c/a ~ 1.5 - 1.6), though the ΦTOP, of course, is much smaller. It is very pleasing to look at. It also feels quite nice in one's hand, producing a sensation something like Chinese "stress relieving" balls do. In some sense the object is reminiscent of Piet Hein's handheld "super-egg" (see Fig. 4a). However, that object is not a good top, though it can stand upright by itself without rotating. The golden ellipsoid, like the super-egg, has a uniquely defined shape – unlike that of an egg (ovoid). Nonetheless, both ovoids and prolate ellipsoids, when spun, can stand upright, provided that the ratio of the major to minor axes lies in the range of about to 1 to 2. What about other values? For a ratio greater than 2, the prolate ellipsoids will not stand erect. What about less than 1? Then one is considering oblate ellipsoids (sort of thick pancakes). These objects can indeed stand upright when spun around their minor axis. What about other physical characteristics of the ΦTOP? If one brings a fairly strong magnet (such as a rare earth neodymium magnet, though an ordinary refrigerator magnet will suffice) near a non-magnetic metal (e.g., aluminum) ΦTOP while it is spinning in the upright position, it will quickly stop spinning. This is a very nice display of induced eddy currents in a moving conductor while in the presence of a magnetic field. It is the inverse of Tesla's "Egg of Columbus" demonstration that spins up a conducting egg using alternating currents.

Super-eggs, Superalls and Golden Ellipsoids

Ellipsoidal forms — including ovoids — have been used as the basis for a number of sculptures over time. Below we show Danish polymath Piet Hein's sculptural "Super-egg" (based on the Lame curve); a "Golden Ellipsoid" made by American artist Randy Rhine (designed with the author); and Canadian artist Gord Smith's "Superall" (which is topologically closely related to the Meissner solid of constant width).



Figures 4a (l.), b (c.) and c (r.): *(a) P. Hein "Super-egg", fiberglass, ~ 4 m tall, 1999; (b) R. Rhine and KB, "Golden Ellipsoid", wood, ~ 13 cm tall, 2015; (c) Gord Smith,"Superall", brass, ~18 cm tall, 1982.*

The Gelatinous Ellipsoid

The great 19th century physicist and philosopher of science Ernst Mach appears to be the first person to have written [3] about a visual phenomenon that can be seen while watching slowly rotating hard boiled eggs: they appear to deform, almost as if they were made of jelly. A two-dimensional version of the effect can be seen on my web site at: http://lite.bu.edu/vision-flash10/applets/Form/Ellipse/Ellipse.html. The ΦTOP exhibits the effect beautifully. The cause of this visual phenomenon is still under investigation.

Summary

This project explored the mathematical, physical, artistic, perceptual and aesthetic aspects of a special prolate ellipsoid that can be called a "golden ellipsoid". Spun rapidly, it is a counter-intuitive top; rotated slowly, it elicits the "gelatinous ellipse illusion"; and it also can be incorporated into elegant sculptures.

Acknowledgments

I thank Boston University engineer Robert Sjostrom for fabricating the brass version of the ΦTOP shown here. I also thank BU engineer David Campbell and BU undergraduate Victor Li for help in making rapid prototype plastic versions of the top. This project is part of a larger study of tops, but also gained impetus from an inquiry about the physics of spheroidal objects from the designer and machinist Richard Berner.

References

[1] K. Brecher, "A Torque About Tops", *Bridges Seoul*, p. 51,Tessellations Publishing, 2014.
[2] M. Livio, *The Golden Ratio*, Broadway Books, 2002.
[3] E. Mach, *The Analysis of Sensations*, p. 234, Dover Pubs., 1959.