Chains of Antiprisms

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Abstract

We prove a property of antiprism chains and show some artwork based on this property.

Introduction

The sculpture *Corkscrew* was featured at the Bridges 2014 Art Exhibition [1]. It is constructed from antiprisms [4] on regular pentagons, which are either stacked together on their pentagonal faces, or connected on their triangular faces to form zigzag chains. In particular, notice the isosceles 'triangles'[1] consisting of a stack of six antiprisms and two zigzag chains of seven antiprisms.

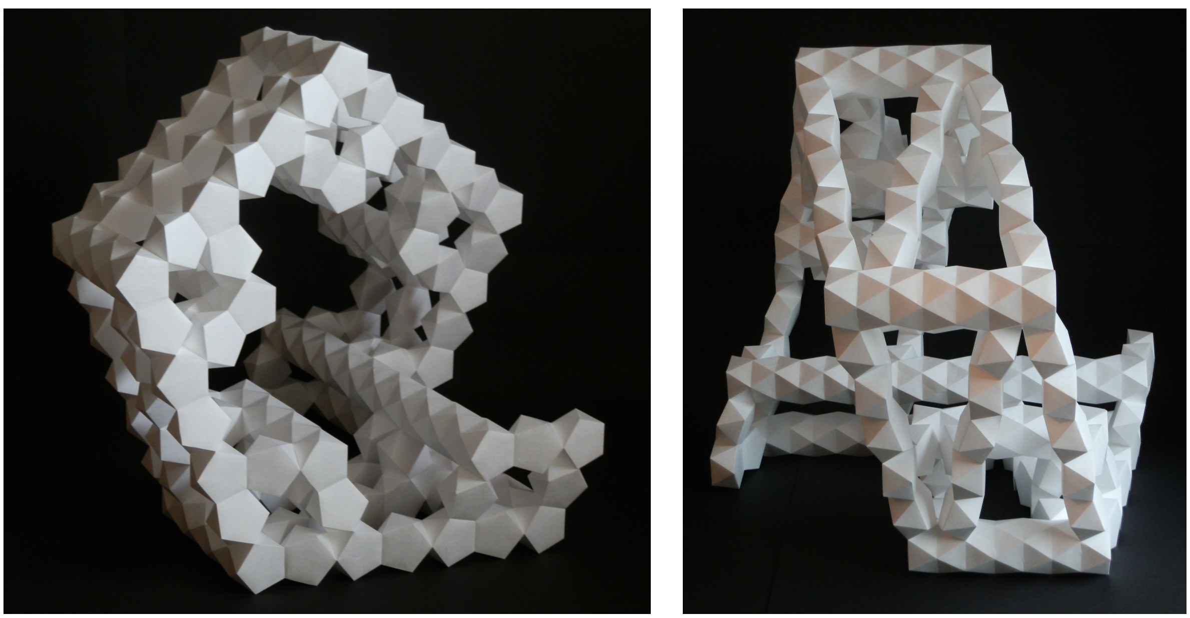

Figure 1 : *Sculpture* Corkscrew *designed and realized by Melle Stoel, constructed from pentagonal antiprisms*

At the time when *Corkscrew* was designed [2], it was only a conjecture that this 'triangle' is mathematically precise, that is, that it does not involve any twisting or bending.

We will prove the following more general theorem. Consider the antiprism on a regular n-gon, for odd n at least 3. It consists of two regular n-gons connected into a polyhedron by $2n$ equilateral triangles (see Fig. 2, left). These antiprisms can be connected via diagonally opposite triangles into a zigzag chain, and can be stacked via the n-gons into a tower (see Fig. 2, right).

[1] There is also a stack of two antiprisms; so, it is more a trapezoid with one quite short edge.

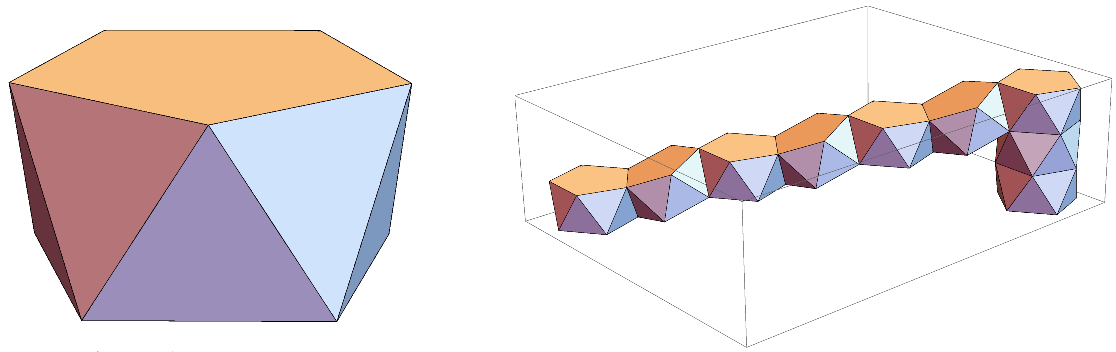

Figure 2 : *Pentagonal antiprism (left); zigzag chain and stack of antiprisms (right)*

Theorem For every odd $n \geq 3$, the diagonal chain of seven such antiprisms is exactly as high as a stack of three antiprisms.

Proof

Consider a regular n-gon with side length 1. Let R and r be the radii of the circumscribed and inscribed circles (see Fig. 3, left). We then have $R \sin(\pi/n) = 1/2$. Furthermore, $R^2 = r^2 + 1/4$ (Pythagoras applied to triangle $R, r, 1/2$). Or, $R^2 - r^2 = 1/4$. Thus $(R - r)(R + r) = 1/4$.

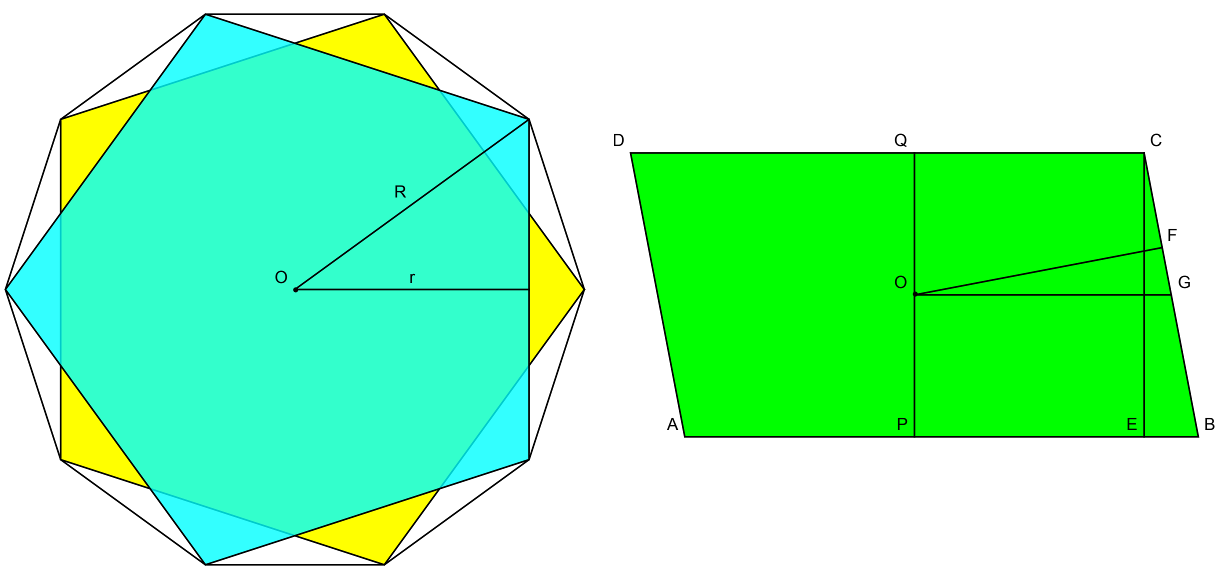

Figure 3 : *Top view (left) and side view (right) of odd antiprism*

In the side view (Fig. 3, right), we see a parallelogram $ABCD$, for which the slanted right edge BC has length $\frac{1}{2}\sqrt{3}$ (the height of an equilateral triangle of side length 1). Its square equals $3/4$. The length of the other (horizontal) edge AB equals $R + r$, since $|PA| = r$ and $|PB| = R$. Note that $|EB| = R - r$. Thus, the height $h = |EC|$ of the parallelogram satisfies $h^2 + (R - r)^2 = 3/4$.

Triangles EBC and FGO are similar (OF is perpendicular to BC; and OG is perpendicular to PQ and to EC). Thus, the lengths of the sides are proportional, that is, for the ratios we have $EB : FG = EC : FO = BC : OG$. We now know

$$\begin{aligned} BC &= \frac{1}{2}\sqrt{3} \\ OG &= (R+r)/2 \\ EB &= R-r \end{aligned}$$

and so

$$\begin{aligned} FG &= EB \cdot OG/BC \\ &= (R-r)(R+r)/\sqrt{3} \\ &= 1/(4\sqrt{3}) \\ &= \sqrt{3}/12 \\ &= BC/6 \end{aligned}$$

From this we find that F is the centroid of the triangular face, and lies at $2/3$ of the height; that is, $1/6$ of the height higher than the centroid of the antiprism.

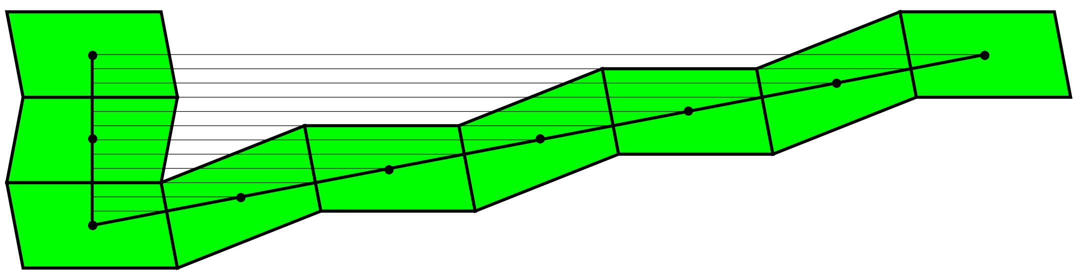

Figure 4 : *Side view of odd antiprism chain, showing how each antiprism center rises one-third of the height*

Because OF is perpendicular to BC, the line OF (when extended) will pass through the centroid of the next (tilted) antiprism in the chain, and from there through the centroid of the third (again horizontal) antiprism. The centroids of consecutive antiprisms in the chain step up $1/3$ of the antiprism's height. A zigzag chain of seven antiprisms rises six times $1/3$ the height, or exactly as much as two antiprisms.

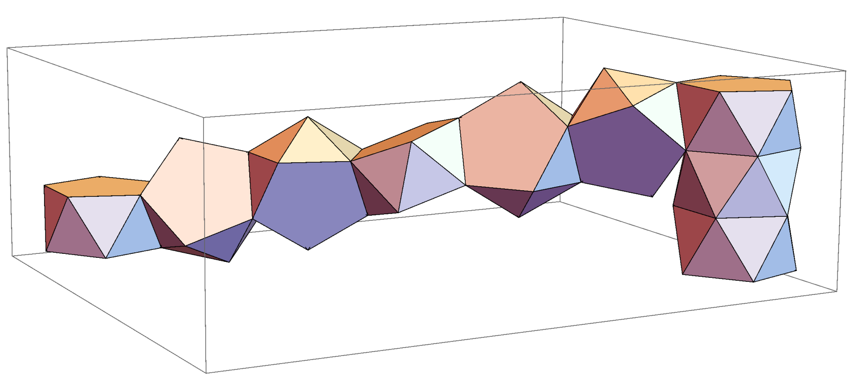

Figure 5 : *Chain of pentagonal antiprisms rotated along the center line*

Corollary

The line through the centroids of the antiprisms in the zigzag chain passes through the centroids of the connecting triangles, intersecting them at a right angle. Consequently, each antiprism in the chain can be rotated over a multiple of 120° along that center line, without disturbing the structure. Figure 5 illustrates this. The triangular antiprism equals the octahedron, which is more symmetric than n-gon antiprisms with $n > 3$. Hence, the rotation of the octahedron in such chains has no visible effect.

This rotational freedom is exploited in the sculpture *Nonagonal antiprism/Hexagonal tesselation*, shown in Figure 6. Here, the straight chains of seven nonagonal antiprisms are bent at every second (i.e., horizontal) antiprism into a triangular shape (three antiprisms to one side), such that the first and the last horizontal antiprism end up above each other, with just room for one antiprism in between them. The connecting non-horizontal antiprisms have been rotated.

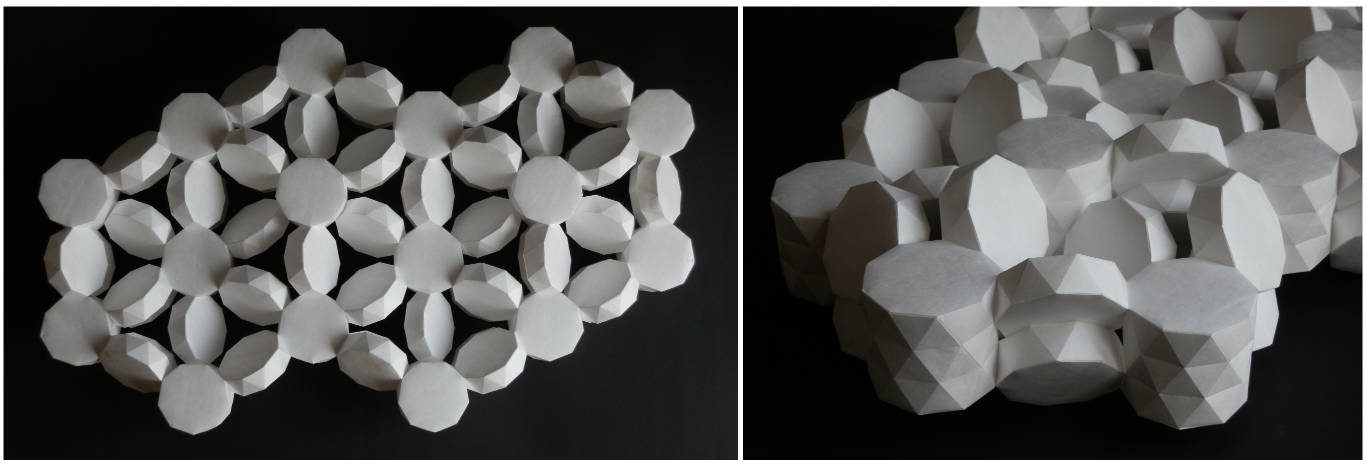

Figure 6 : *Nonagonal antiprism/Hexagonal tesselation, by Melle Stoel*

Related Work In [3], Stewart studied various loops of polyhedra attached face to face. We do not know whether this includes loops of antiprisms.

References

[1] Melle Stoel. *Corkscrew*. Bridges 2014 Art Exhibition. `http://gallery.bridgesmathart.org/exhibitions/2014-bridges-conference/mellestoel` (accessed 15 Mar 2015).

[2] Melle Stoel. *Closed Loops of Antiprisms*. *Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture*, pp.285–292.

[3] B. M. Stewart. *Adventures Among the Toroids: A Study of Quasi-Convex, Aplanar, Tunneled Orientable Polyhedra of Positive Genus Having Regular Faces With Disjoint Interiors* (2nd Ed.). Self-published, 1980.

[4] Wikipedia contributors. *Antiprism— Wikipedia, The Free Encyclopedia*. `https://en.wikipedia.org/wiki/Antiprism` (accessed 15 Mar 2015).