Eight-Pointed Star and Precise Construction of 7×7 Square Grid

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Abstract

The paper explores two methods of constructing precise 7×7 square grid based on the eight-pointed star by Tons Brunes. Both methods are closely connected with the ancient proof of the Pythagorean theorem for 3,4,5 right triangle and *ad quadratum*.

Eight-Pointed Star by Tons Brunes

In the reconstruction of ancient geometry by Tons Brunes a key role belongs to the square with an eight-pointed star inscribed in it [1]. The star is composed of eight straight lines connecting the vertices of the square with the middle points of its sides.

The important property of the Brunes star is that its lines generate 3,4,5 right triangles known in ancient times also as "Egyptian" or "sacred" triangles (fig. 1, *a*). Brunes demonstrated that points of intersection of the star lines can be used to divide the sides of the square into 2, 3, 4 and 5 equal parts and the square can be transformed into regular grids (fig. 1, *b*). On the base of these basic divisions Brunes constructed 6×6, 8×8, 9×9 and 10×10 regular square grids. His only exception was the 7×7 grid.

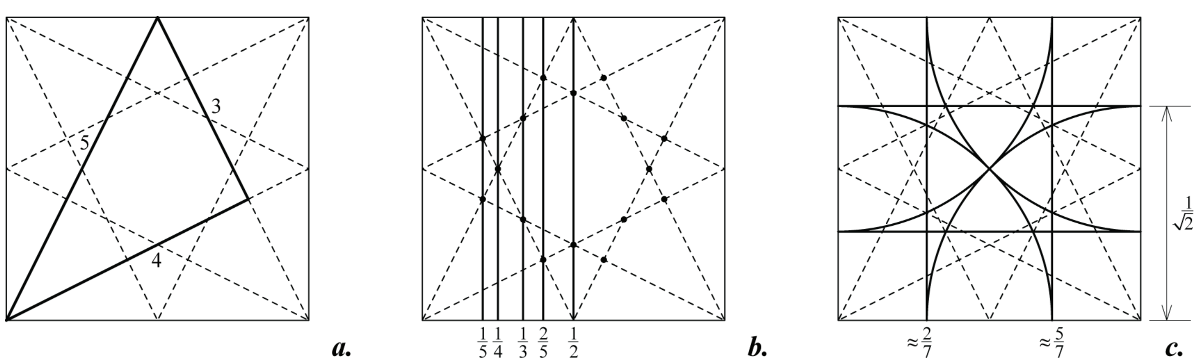

Figure 1: *Brunes Star and 3,4,5 right triangle (a) and division of square into 2, 3, 4, and 5 equal parts (b) and approximately 7 equal parts with "sacred cut" construction (c).*

To get the 7×7 grid Brunes used the fact that half the diagonal of the unit square, 0.7071..., approximates 5/7 of its side. Four arcs centered in the vertices of the square and with the radii of 1/2 of the square diagonal construct what Brunes named a "sacred cut" (fig. 1, *c*). In a sacred cut the division of the side of square into 7 approximately equal parts is combined with rectifying the circle circumference, because the length of each diagonal of half the square that constitutes the eight-pointed star is approximately equal to the length of arc with radius of 1/2 of the square diagonal. According to Brunes, perimeter of the 7×7 square consisted of 28 parts symbolized to the Ancients 28 days of moon month.

Division with Lines Nonparallel to Square's Sides

It may seem that the only possible method to divide the side of square into equal parts is to draw straight lines parallel to the sides of square through the points of intersection of the star. But there are some pairs of points that give precise divisions of the sides of square, even though lines drawn through these points are nonparallel to them.

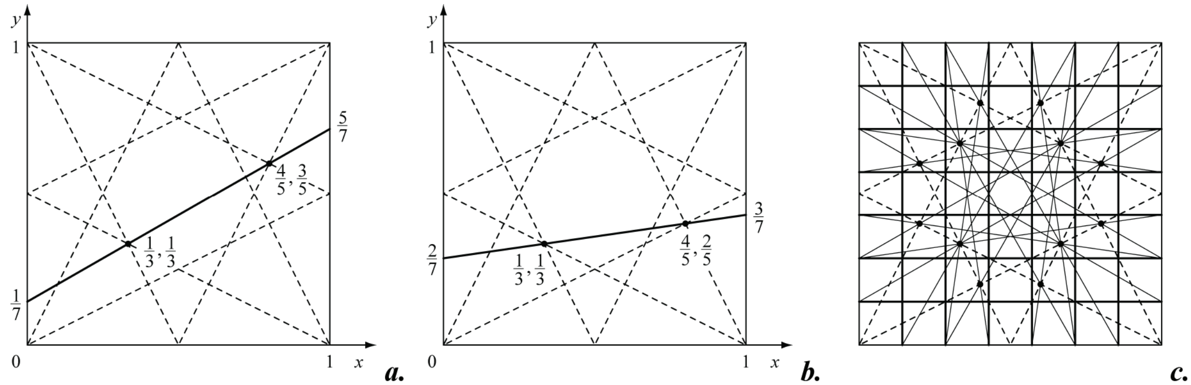

Figure 2: *Lines of two types (a, b) that pass through some points of intersection of the Brunes star and are nonparallel to the sides of square divide them into 7 equal parts (c).*

Place a unit square with lower left corner at the origin of Cartesian coordinates. The line of the first type passes through points of intersections of the Brunes star (1/3, 1/3) and (4/5, 3/5) (fig. 2, *a*). Its equation is

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{4}{7} x + \frac{1}{7}.$$

The line intersects the sides of square at the points (0, 1/7) and (1, 5/7), from which the vertices of the 7×7 grid can be constructed. The line of the second type passes through the points (1/3, 1/3) and (4/5, 2/5) (fig. 2, *b*). Its equation is

$$y = \frac{1}{7} x + \frac{2}{7}.$$

The line intersects the sides of square at the points (0, 2/7) and (1, 3/7), again permitting a 7×7 division to be constructed. The symmetrical reflections of the lines in four axes of square give the system of 16 lines that precisely divides each of the sides of square into 7 equal parts (fig. 2, *c*)

The continued line of the second type crosses with the central horizontal axis of symmetry of the star at the point (3/2, 1/2). This point is the vertex of the square with the side length of √2 circumscribed around the unit square (fig. 3, *a*). The most common way to construct a square of double area is to continue the arcs of the sacred cut outside of the 1×1 square perimeter to their intersections which give the vertices of the √2×√2 square (fig. 3, *b*).

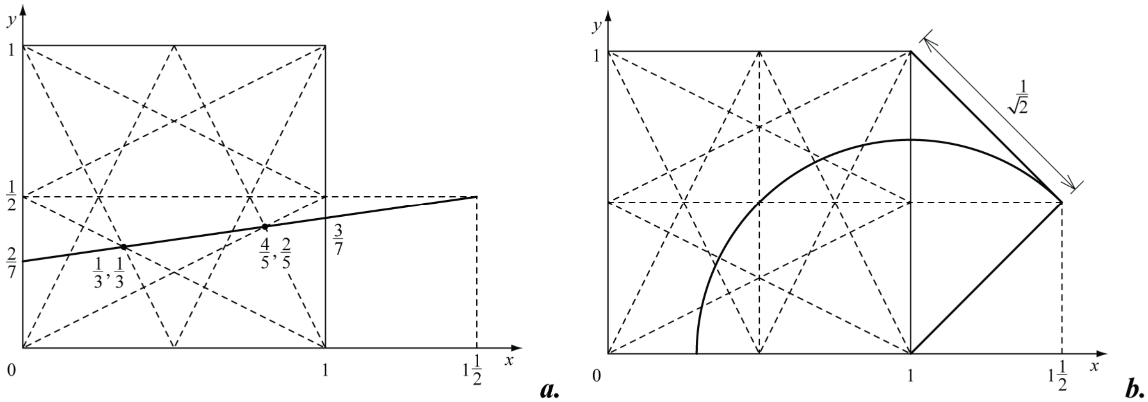

Figure 3: *Doubling square with continued lines of division of square into 7 equal parts (a) and with continued arcs of the Sacred Cut (b).*

Doubling the square was known in ancient architecture as *ad quadratum*. The precise 7×7 square grid together with the Brunes star gives an alternative method of doubling the square without using any elements of circles. For practical purposes this method is much more useful because the drawing of large-scale straight lines is not dependent of extensibility of measuring ropes as opposed to the drawing of arcs that demands constant rope stretching.

Pythagorean Theorem for the 3,4,5 Triangle

To construct the √2×√2 square it is sufficient to draw only four lines of the second type and elongate them to the intersections with the vertical and horizontal axes of the star (fig. 4, *a*). The 7×7 grid in the unit square makes it possible to draw another four lines parallel to the four original ones that pass through the points of intersections of the star. The expansion of this new modular grid to fill up the √2×√2 square gives the natural measuring lines for the right triangles with hypotenuses on the sides of the √2×√2 square and the legs lie on the lines of the second type. It is obvious that they are 3,4,5 right triangles and their legs have exactly 3 and 4 lengths of the modular grid (fig. 4, *b*). The completing of the 3,4,5 right triangles to 3×4 rectangles transforms the drawing to the well known proof of the Pythagorean theorem (fig. 4, *c*).

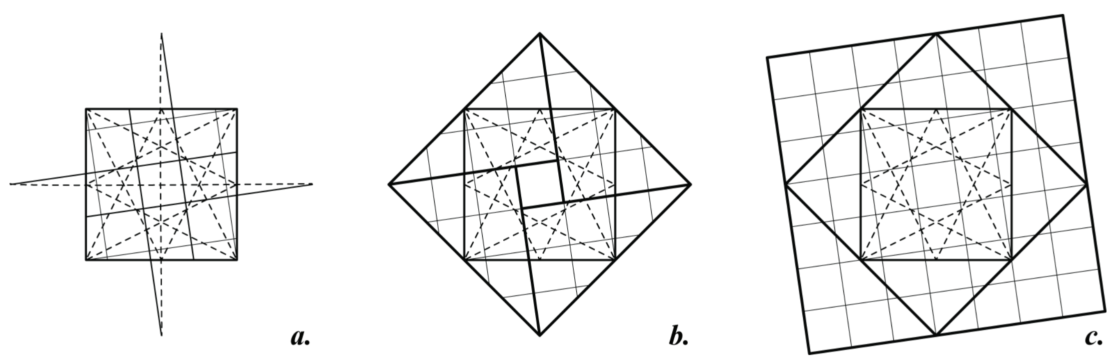

Figure 4: *Doubling square by elongation of lines of division of square (a) and getting four 3,4,5 right triangle (b) that is the base for ancient proof of the Pythagorean theorem (c).*

It is possible to hypothesize that this geometric construction represents the real origin of the Brunes star. It combines the 7×7 square grid naturally received by adding together eight 3,4,5 right triangles and the construction of *ad quadratum* on the base of a 5×5 square with eight diagonals of half the square 5/√2×5/√2 inscribed in it which also generates 3,4,5 right triangles. As a result it becomes visually obvious that four central lines of the 7×7 grid intersect with points of the Brunes star.

Doubling the Square and Doubling the Star

Construction of doubling the square together with the Brunes star inscribed into the inner square suggests that we might inscribe a second Brunes star into the outer square as well. These two stars have common points of intersections that make it possible to divide the sides of the inner square into 7 equal parts with lines parallel to its sides.

The lines of the outer star divide the sides of the inner square into three equal parts (fig. 5, *a*). Two pairs of lines that belong to the inner and outer stars and start at the vertices of the inner square – points with coordinates (0,0) and (1,0) – intersect the opposed sides of the inner square at the points (1/3,1) (1/2,1) and (0,1/3) (0,1/2) correspondingly. These pairs of lines have the equations:

$$y = 3x, \quad y = 2x \quad \text{and} \quad y = -\frac{1}{3}x + \frac{1}{3}, \quad y = -\frac{1}{2}x + \frac{1}{2}.$$

The lines of these two pairs that belong to different stars intersect in the points with coordinates (1/7,2/7) and (1/7,3/7). Four similar pairs of lines give 16 points of intersection that determine all the necessary elements to transform the inner square to an exact 7×7 grid (fig. 5, *b*).

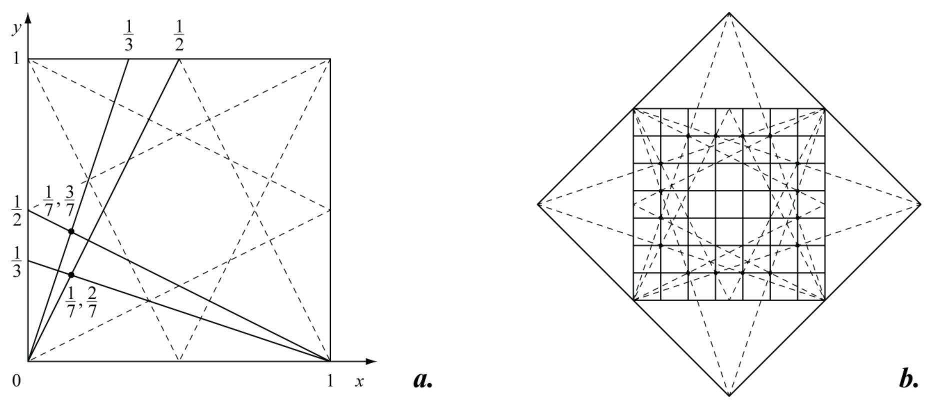

Figure 5: *Two Brunes stars inscribed into both squares of ad quadratum construction give points of intersection (a) that permits dividing the sides of inner square into 7 equal parts with lines parallel to them (b).*

Conclusion

We have avoided most analytic methods in this paper because of space limitations, and to emphasize the visual proofs. All these results may be obtained just by means of the similarity of triangles well known in ancient times.

The eight-pointed star introduced by Tons Brunes about 50 years ago as the base figure of ancient secret and sacred geometry continues to inspire modern researchers [2, 3]. The finding of the missing 7×7 grid inside and outside of this figure reveals its close connection with the Pythagorean theorem for 3,4,5 triangle as well as geometrical constructions of doubling the square and *ad quadratum*.

References

[1] T. Brunes, *The Secrets of Ancient Geometry and its Use*, Rhodos, Copenhagen, 1967.
[2] J. Kappraff, *Beyond Measure*, World Scientific, Singapore, 2001.
[3] J. Kappraff, *The Sacred Cut*, Proceedings of Bridges: Mathematics, Music, Art, Architecture, Culture, pp. 559 – 562, 2011.