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The Musical Canon Inside Differential Equations

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Abstract

A musical canon contains two or more lines of music, each playing the same basic melody, but with their starting points shifted. Surprisingly, exactly this same structure emerges among an interesting set of differential equations (including some that are important for physics) that exhibit the property known as *shape invariance*. In this paper, I demonstrate the parallelism of these structures.

Introduction

Imagine listening to a melody. Perhaps it is appealing, perhaps not. But now, for some reason, you listen to two people playing the same melody, except one has started a measure after the first person. For most melodies, the result would be unremarkable at best, and more likely somewhat cacophonous. But for certain melodies, something surprising emerges: a euphonious piece of music, the two lines harmonizing effectively with each other. Not only does the second measure suitably follow the first measure melodically, but also, at the same time, the second measure meshes harmonically with the first measure. Until you played the melody in this dual way, this structure was hidden, but it was clearly essential to how the piece was composed.

Of course, the structure could be more elaborate. Three or four people could play the melody, each entering a fixed interval after the previous voice, and they might still fit together. Or perhaps the second line is shifted in time and in pitch, say up a fifth, and, if there are additional voices, these might each come in a fifth above the previous voice. And while most melodies would yield something akin to noise when played this way, there are a select few melodies that yield something harmonious and appealing. The pieces produced by such self-harmonizing melodies are called canons.[1]

Discovering that a melody is self-harmonizing reveals a structure underlying the melody that is not apparent unless the melody is paired with itself in the appropriate way. Surprisingly, a very similar structure emerges among a class of differential equations, including some that are important for physics.

In particular, we will look at differential equations which we treat as eigenvalue problems. As a toy model, we might ask, "For what functions f is $-\frac{d}{dx}f(x) = Ef(x)$, where E is a constant?" To make this better defined, we might require that the functions be defined on the positive real axis, and go to zero as $x \to \infty$. The functions that satisfy this condition are $f_a(x) = Ne^{-ax}$ where a is a positive real number, and $E = a$. Thus, the possible values of E are the positive real numbers; this is called the spectrum of $-\frac{d}{dx}$. The functions $f_a(x)$ are the eigenfunctions of $-\frac{d}{dx}$, and we can choose some convention (e.g., $f(0) = 1$) to determine a unique eigenfunction for each value in the spectrum.

We can generalize this problem to any differential operator, that is, an expression like $-\frac{d^2}{dx^2} + \gamma x^4$ or $-\frac{d^4}{dx^4} + 3x\frac{d^3}{dx^3} + \cosh(x)$. When we do so, we can typically show that there is some spectrum and set of

[1] One should be aware that in most actual canons, composers include slight modifications between the voices, although this is not the case in a perfect round, for example.

eigenfunctions, but without a way to determine these; even knowing part of the spectrum or some of the eigenfunctions does not tell you what the rest of the spectrum and eigenfunctions are.

However, there are specific differential operators—appearing in Sturm-Liouville problems and quantum physics—for which the exact spectrum and eigenfunctions can be found, using methods from analysis. Why these operators were special was not clear. But in work over the past half century [8] [5] [2] [4] [11], it has emerged that the spectrum and eigenfunctions of any one of these special differential operators is not properly understood alone. Rather, each of these exactly solvable differential operators has a partner differential operator (obtained via supersymmetry) such that, for example, the second eigenfunction of one differential operator must relate, in a specific way, to the first eigenfunction of its partner differential operator. The partner differential operator can be obtained in two distinct ways from the original differential operator, and the reason that these differential operators have spectra and eigenfunctions that can be solved exactly is because of the two simultaneous ways they must pair with their shifted partners. Just as, in a canon, the need to harmonize effectively with the first measure determines the form of the second measure, the need to be a supersymmetry transform of the first eigenfunction determines the form of the second eigenfunction.

In short, discovering the algebraic reason these particular problems are analytically exactly solvable is like discovering that a melody you had known all your life was actually designed to be part of a canon.

In writing for an audience with mixed expertise, it is a challenge to choose between the general formulation and specific examples. The simplest specific example does exhibit the necessary features, but at the same time, does not exhibit the full possible structure. But the general formulation requires more background than it is feasible to develop in this paper. Thus, I will focus primarily on a simple example, but then also discuss what can be generalized.

This paper was inspired by Noam Elkies' public lecture at Bridges 2014, and the recognition that the musical structures he was describing had the same form as the mathematical structures I was familiar with from the study of shape invariant differential equations in quantum mechanics

Canons in Music

In music, a canon is a piece in which there are several lines, each of which plays the same melody, but starting at different times [9]. In the simplest case, one has two voices, each playing the exact same melody, but with the two voices shifted in time, the second voice starting after the first voice has already played several notes.

Even within the canon form, there is a richness of possibilities. One might have two, three, four, or more voices. There are canons in which each later entering voice is transposed relative to the previous voice; canons in which voices may appear reflected in pitch (e.g., the second voice plays the melody of the first voice in contrary motion, one melody going up in pitch where the other goes down); and canons in which the duration of notes in the voices might be subject to an overall scale factor (e.g., the second voice plays the same melody as the first voice, but each note lasts twice as long as its counterpart in the first voice).

For our purposes, we can focus primarily on the so-called *simple canon*, in which each voice plays exactly the same melody as the first voice, at the same pitches, in the same tempo, and furthermore, with each voice entering by the same time delay from the previous voice. For the most part, it will suffice to look at a two voice simple canon, discussing generalizations as appropriate. We will denote the N^{th} measure of the overall piece as *measure N*, and the N^{th} measure of the basic melody as *segment N*.

For convenience, we make the further restrictions that the measures of the piece are rhythmically regular (i.e., each measure has the same number of beats) and that each voice comes in one measure after the previous voice. Thus, the second voice enters when the first voice begins its second measure; if there were are voices, then in the ninth measure of the piece, the first voice plays the ninth segment of the melody at the same time that the second voice plays the eighth segment, the third voice plays the seventh segment, and the fourth voice plays the sixth segment. Letting the time delay between voices be one measure gives us an easy way to speak of the time shift between different lines.

How might one write a two-part simple canon? One determines segment #1 of the melody and places this in measure #1 of the first voice. Measure #1 of the second voice is left empty, but then measure #2 of the second voice is filled with segment #1 of the melody. Using the rules of counterpoint, we now determine segment #2 of the melody to place in measure #2 of the first voice, so that it fits appropriately with what else is being played at that time (namely, segment #1 in voice two). Now segment #2 is placed in measure #3 of the second voice; the rules of counterpoint let us determine a segment #3 of the melody, which goes in measure #3 of the first voice, as well as in measure #4 of the second voice. One continues in this way, with the rules of counterpoint allowing us to pair what goes on in measures played simultaneously, and the time shift of the canon letting us move forward from measure N in voice one to measure $N+1$ in voice two.

Canons in Differential Equations

Sometimes one studies a differential equation with a particular set of initial conditions so that there is a unique solution. However, there are also situations that arise—this is, for example, often the case in physics or when solving a partial differential equation by means of separation of variables—in which one has an ordinary differential equation of a form such as

$$-\frac{1}{2}\frac{d^2}{dx^2}\phi(x)+U(x)\phi(x)=E\phi(x) \tag{1}$$

where $U(x)$ is specified, but we are looking for functions $\phi(x)$ and constants E that together provide a solution to the equation. We can organize the solutions as $\{E_1,\phi_1(x)\}$, $\{E_2,\phi_2(x)\}$, $\{E_3,\phi_3(x)\}$, ..., where we have $E_1<E_2<E_3<\cdots$. We will refer to the E_k as *energies*, since that is the role they play in quantum mechanics, but this identification is not essential; here it is just a convenient way to refer to these constants.

Most such systems cannot be solved exactly, but there are select examples whose full spectrum can be determined exactly by power series or other methods of analysis. The reason such solutions exist is not apparent; it just seems like a mathematical coincidence.

But in [8] [5] [2] [4] [11], the underlying reason these particular systems can be solved exactly has been identified. One finds that in those situations, system (1) has a partner equation, to be described below. The partner equation has its own solutions $\{\widetilde{E}_1,\widetilde{\phi}_1(x)\}$, $\{\widetilde{E}_2,\widetilde{\phi}_2(x)\}$, $\{\widetilde{E}_3,\widetilde{\phi}_3(x)\}$, ... However, in the exactly solvable cases, the solutions of the partner equation are related to those of the original equation by $\widetilde{E}_k=E_{k+1}$ and $\widetilde{\phi}_k(x)=\phi_k(x)$. This relationship is shown in Figure 1.

Once we establish this structure, the differential equation almost solves itself, in a way analogous to what happens in a musical canon. The idea is that the original equation has a solution with energy E_1 with function $\phi_1(x)$. The partner equation has no solution with energy E_1. However, the function $\phi_1(x)$ *is* a solution of the partner equation with energy E_2, while the original equation has a different function $\phi_2(x)$ that has energy E_2. This function $\phi_2(x)$ appears as the function with energy E_3 in the solution to the partner equation, but then one finds a separate function $\phi_3(x)$ that solves the original equation with energy E_3.

The parallel to the musical canon should be clear. For a given energy value E_k, the original equation has a solution $\phi_k(x)$ while the partner equation has a solution $\phi_{k-1}(x)$. This is what happens in a two-part simple canon, where at the k^{th} measure of the piece, the first voice plays segment k of the melody while the second voice plays segment $k-1$. Thus, the first line of the canon is like the original equation, and the second line of the canon is like the partner equation. Each new energy corresponds to a new measure. In

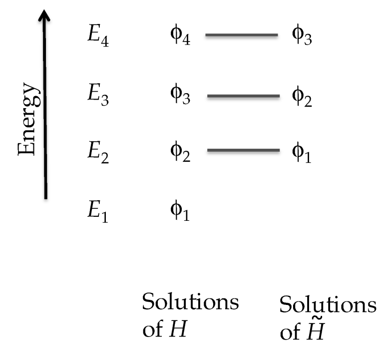

Figure 1 : *The solutions of the H and $\widetilde{H}$ equations*

the canon, whatever appears in measure k for the first voice appears in measure $k+1$ for the second voice, so that at measure $k+1$, one will hear segment $k+1$ and segment k; in the differential equation, whatever appears with energy E_k for the original equation appears with energy E_{k+1} for the partner equation, so that at energy E_{k+1} there will be two functions, $\phi_{k+1}(x)$ and $\phi_k(x)$, with that same energy. With the original and partner differential equations, there will be a mathematical rule allows us to move between the two differential equations, just as counterpoint allows us to move between the two lines of the canon (although the rules of counterpoint do allow for more flexibility than the mathematical correspondence does).

Differential Equations: Spectrum

We now proceed to find the canon hidden within some differential equations. For the purposes of this paper, we will consider second-order ordinary differential equations of the form

$$-\frac{1}{2}\frac{d^2}{dx^2}\phi(x)+U(x)\phi(x)=E\phi(x) \tag{2}$$

where we are trying to determine functions $\phi(x)$ and real constants E that will satisfy this equation. (To keep things well-behaved, we typically only seek $\phi(x)$ that are square integrable.) Solving this problem is essential in quantum mechanics, where this equation is called the time-dependent Schrödinger equation and where E corresponds to the physical energy [7], but this sort of problem arises more generally when trying to find sets of basis functions in a large variety of contexts.

One way to look at the equation (2) is as an eigenvalue problem. We define the operator

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+U(x) \tag{3}$$

and then our initial differential equation (2) can be understood as finding solutions to the equation

$$H\phi(x)=E\phi(x) \tag{4}$$

Thus, the problem we have before us can be understood as finding the eigenfunctions $\phi_k(x)$ and corresponding eigenvalues E_k of the object H. This is known as finding the spectrum of the differential operator H.

Fortunately, for those unfamiliar with the above terminology, the actual math that we will need is merely that of ordinary one variable calculus.

Differential Equations: Spectrum Pairing

Let us suppose that we are trying to solve $H\phi(x) = E\phi(x)$, but we find that there is a way to factorize H and write $H = AB$. For right now, we will just examine this problem formally, but in the next section, we will see a particular example of this structure. (Although the order of A and B matters, the order of A or B and any real number like E can be interchanged at will.)

So if we have A and B such that $H = AB$, what happens if we consider $\widetilde{H} = BA$? What is the connection between H and $\widetilde{H}$? The answer is that whenever H has a solution with energy E, so does $\widetilde{H}$, and vice versa. The functions need not be the same, but the values of E are.

To see this, let us imagine we have a solution to $H\phi = E\phi$, or, in other words $(AB)\phi = E\phi$. Then

$$EB\phi = B(E\phi) = BH\phi = B(AB)\phi = (BA)B\phi = \widetilde{H}(B\phi) \tag{5}$$

If we look at this chain of equality, we see, then, that if ϕ satisfies $H\phi = E\phi$, then $B\phi$ satisfies $\widetilde{H}(B\phi) = E(B\phi)$. A similar argument shows that if $\widetilde{H}\psi = E\psi$, then $H(A\psi) = E(A\psi)$.

In other words, energies E that we obtain for $H = AB$ are the same as the ones we obtain for $\widetilde{H} = BA$. Associated with this is also a pairing of the functions that give a common E, by acting on that function with either A or B, depending on which direction we are going. (The origin of this pairing is supersymmetry [2]).

Note that there is one caveat in the above. If there is a function $B\phi_0 = 0$, it *does* satisfy $H\phi = E\phi$ with energy $E = 0$, but there is no corresponding function $B\phi$ to be a solution to the $\widetilde{H}$ equation. (A similar statement holds if $A\psi_0 = 0$.) Thus we need to modify our statement: the pairing of solutions for the H and $\widetilde{H}$ equations holds for the non-zero energies only. This actually will be beneficial to us!

This pairing has gotten us halfway to our goal, associating solutions to the H differential equation and to the $\widetilde{H}$ differential equation by pairing solutions with the same E value, much as we can associate, in a canon, any two measures played at the same time. But this is just the first step. It is not until we get to the next property that the canon structure becomes both apparent and natural.

The Quantum Harmonic Oscillator

We now turn to a particular example, namely the equation

$$-\frac{1}{2}\frac{d^2}{dx^2}\phi(x) + \frac{1}{2}(x^2 - 1)\phi(x) = E\phi(x) \tag{6}$$

This simple equation is not only intimately associated with the Hermite polynomials [1], but it is also of fundamental importance in quantum physics. It is the equation for the quantum harmonic oscillator, appears in the mathematical derivation that shows that electromagnetic and other fields can be described in terms of particles, is used to describe vibrations of molecules, and plays a key role in spectroscopy.

We can write equation (6) as $H\phi(x) = E\phi(x)$, where

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(x^2 - 1) \quad . \tag{7}$$

We will define

$$A = \frac{1}{\sqrt{2}}\Big(-\frac{d}{dx} + x\Big) \qquad\qquad B = \frac{1}{\sqrt{2}}\Big(\frac{d}{dx} + x\Big) \tag{8}$$

Then, it is a simple calculation to see that for any function $f(x)$,

$$A(B(f(x))) = A(Bf(x)) = \frac{1}{2}\Big(-\frac{d}{dx}+x\Big)\Big(\frac{df(x)}{dx}+xf(x)\Big) = -\frac{1}{2}\frac{d^2}{dx^2}f(x)+\frac{1}{2}(x^2-1)f(x) = Hf(x) \quad (9)$$

and so $H = AB$. Using the results of this previous section, this means that energies E for which H has solutions are also the energies E for which $\widetilde{H}$ has solutions. What is $\widetilde{H}$? Calculating, we see

$$\widetilde{H}f(x) = B(Af(x)) = -\frac{1}{2}\frac{d^2}{dx^2}f(x)+\frac{1}{2}(x^2+1)f(x) \quad , \quad (10)$$

and so $\widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}(x^2+1)$.

What is striking about this is that $\widetilde{H} = H+1$. In other words, other than a shift by an overall constant, $\widetilde{H}$ and H are the same. This allows us to find solutions to the differential equation.

Canons and the Solution to the Harmonic Oscillator

We are now in a position to find the solutions (energies and eigenfunctions) to $H\phi(x) = E\phi(x)$. What do we know so far? In addition to the general properties connecting H and $\widetilde{H}$, we now have the additional result that $\widetilde{H} = H+1$. Thus, if $H\phi = E\phi$, $\widetilde{H} = (E+1)\phi$. This means that the solutions to the H and $\widetilde{H}$ equations are paired in an even tighter structure than the simple *AB vs. BA* relationships would imply.

Recall that from the general pairing of $\widetilde{H}$, we know that if $H\phi_k = E_k\phi_k$, then $\widetilde{H}\phi_k = E_{k+1}\phi_k$. From the particular result $\widetilde{H} = H+1$, we can go further: $\widetilde{H}\phi_k = (H+1)\phi_k = (E_k+1)\phi_k$. Put all together, this yields

$$E_{k+1} = E_k + 1 \quad (11)$$

So now how do we solve the original equation? To get started, we need one technical point. As is shown in the appendix, the energies for $H\phi = E\phi$ for (6) have to be non-negative. Thus, if we can find a solution to $H\phi = 0$, this has to correspond to E_1, the lowest energy in the spectrum. Furthermore, since $\widetilde{H} = H+1$, this means the lowest possible energy for the $\widetilde{H}$ equation is 1.

Finding a solution to $H\phi_1 = 0$ is easy, as we need simply solve $B\phi_1(x) = 0$. The solution is obtained via

$$0 = B\phi(x) = \frac{1}{\sqrt{2}}\Big(\frac{d}{dx}+x\Big)\phi(x) \qquad \Longrightarrow \qquad \frac{d\phi_1(x)}{dx} = -x\phi_1(x) \quad (12)$$

from which it follows quite straightforwardly that $\phi_1(x) = e^{-x^2/2}$. (Since we are thinking of these solutions as eigenfunctions or as basis functions like the sines and cosines of Fourier theory, multiplying this solution by an overall constant still produces a solution, but does not change the meaning or significance of the solution.)

With $\phi_1(x)$ in place, we now work our way up the paired equations, in what looks just like a two-part simple canon. We proceed as follows:

- With energy $E = 0$, H has a solution $\phi_1(x) = e^{-x^2/2}$.
- There is no $\widetilde{H}$ solution with energy 0.
- With energy $E = 1$, $\widetilde{H}$ has a solution $\phi_1(x)$, while H has a solution $\phi_2(x) = A\phi_1(x)$.
- With energy $E = 2$, $\widetilde{H}$ has a solution $\phi_2(x)$, while H has a solution $\phi_3(x) = A\phi_2(x)$.
- The solution for H at energy N (where N is a natural number) shows up at energy $N+1$ for $\widetilde{H}$.

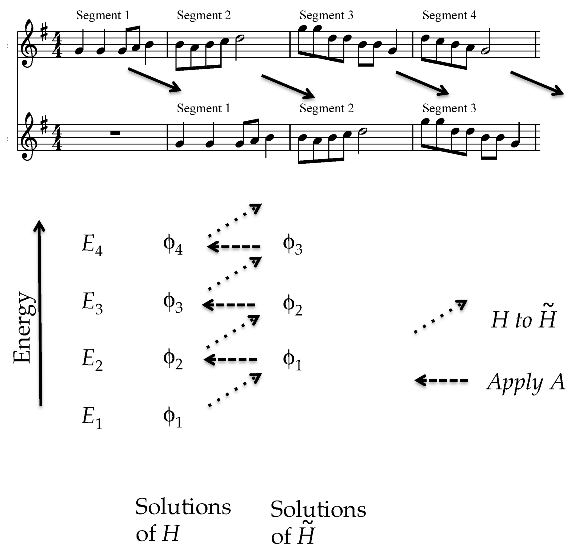

Figure 2 : *The canon structure in music and differential equations.*

- In general, if the $\widetilde{H}$ equation has a solution ϕ_N at energy $N+1$, the H equation has a solution $A\phi_N$ at that same energy $N+1$; we define $\phi_{N+1} = A\phi_N$.

Thus, we see that the analogy to the structure of a two-part simple canon is complete. In the canon, segments N and $N+1$ are played during measure $N+1$, just as in the harmonic oscillator solutions ϕ_N and ϕ_{N+1} appear at energy $N+1$. In the canon, the segment that appears in voice one at measure N is the same segment that appears in voice two at measure $N+1$, just as the solution that appears at energy N for the H equation appears at energy $N+1$ for the $\widetilde{H}$ equation. This is summarized compactly in Figure 2.

Conclusion and Generalizations

The analogy I have described above is very neat and tidy. Is this just a special case, or is this applicable in a more general context? The answer is the latter: this is not just a statement about two-part canons and the quantum harmonic oscillator, but can be applied in a wider array of circumstances.

First, the generalization to multi-part canons emerges naturally from the description above. For simplicity, consider still the quantum harmonic oscillator. We showed that the solutions of H and $\widetilde{H}$ were related like the measures of a two-part musical canon because $\widetilde{H} = H+1$. It is a simple observation to recognize that if H and $\widetilde{H} = H+1$ are related like two voices in a musical canon, then we can iterate this process and created a third object $\hat{H} = \widetilde{H}+1 = H+2$ which is partnered with $\widetilde{H}$ from the other direction: the $\widetilde{H}$ solutions, which appear one energy higher for $\widetilde{H}$ than for H, will appear one energy lower for $\widetilde{H}$ than for $\hat{H}$. Thus, we can iterate our process to create the analogue of a musical canon with three, four, or more voices.

Second, the harmonic oscillator equation is not the only shape invariant equation. Others are well-known and important in the physics literature, including [2] the radial equation for the hydrogen atom, the particle in a box, the Poschl-Teller potential, as well as models which offer string theory-motivated generalizations of quantum mechanics [10] and the multi-particle Calogero model [3] [6]. Indeed, exactly solvable systems in quantum physics seem quite generally to exhibit shape invariance.

To understand these generalizations, note that systems can be shape invariant without having integerspaced energies. Instead of a relationship as simple as $\widetilde{H} = H + 1$, the relationship can be slightly more complicated. Let H depend on some real number g (a parameter in the differential equation). Then the pairing of AB and BA still works as described above, and we still say the system is shape invariant when $\widetilde{H}$ and H have the same mathematical form. However, the parameter g allows some extra flexibility. For example, in some cases we will find that $\widetilde{H}(g) = H(g+1) + c(g)$, where $c(g)$ is a real number constant that depends on g. (Note that $H(g+1)$ is just one example; other functions of g could appear, depending on the system at hand.) The $c(g)$ plays the role here that shifting the energy by 1 unit did in the harmonic oscillator. The appearance of $H(g+1)$ indicates that the partner equation has the same form as $H(g)$, but its eigenfunctions that appear in the solutions are rescaled in some way. Thus, we are still encountering the structure of a musical canon, but the kind in which each new voice is a transformed form of the preceding voice.

Appendix

To see that the energies in (6) are non-negative, recall that $H = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)\frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)$, and use integration by parts to get

$$\int_{-\infty}^{\infty} \phi(x) H \phi(x) dx = \frac{1}{2}\int_{-\infty}^{\infty} \Big(\frac{d\phi(x)}{dx} + x\phi(x)\Big)\Big(\frac{d\phi(x)}{dx} + x\phi(x)\Big) dx \tag{13}$$

Since $\phi(x)$ must be square integrable, it vanishes at $x = \pm\infty$. The expression on the right is the integral of a function squared, and so must be non-negative. Thus, if $H\phi(x) = E\phi(x)$, we have

$$E\int_{-\infty}^{\infty} \Big(\phi(x)\Big)^2 dx = \int_{-\infty}^{\infty} \phi(x) H \phi(x) dx = \frac{1}{2}\int_{-\infty}^{\infty} \Big(-\frac{d\phi(x)}{dx} + x\phi(x)\Big)^2 dx \geq 0 \tag{14}$$

Since the integral of $\phi^2(x)$ over all space must also be non-negative, this implies that $E \geq 0$.

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