The Stomachion in Wonderland

S J Spencer
The Sycamores, Queens Road, Hodthorpe
Worksop, Nottinghamshire
S80 4UT, England
pythagoras@bcs.org.uk
www.grandadstan.com

Abstract

The Walter's Art Museum, Baltimore, has a copy of some work attributed to Archimedes. This includes the description of a Tangram type puzzle, the *Stomachion,* in the form of a dissected rectangle. This year, 2015, is the 150th anniversary of the publication of *Alice in Wonderland* by Charles Dodgson. Dodgson was a Maths Professor at Oxford University, UK. He was interested in the Tangram and wrote literature under the name of Lewis Carroll. This paper examines the geometric properties of a general version of the Stomachion as a parallelogram. A range of designs are created. Examples from *Alice in Wonderland* and *Through the Looking Glass* are used to decorate the designs and fractals. The stories are based, loosely, on a pack of cards and a chess set. The shapes from a general Stomachion are used to create designs for a pack of cards and a chess set.

Charles Dodgson

Charles Dodgson, best known as Lewis Carroll, wrote *Alice in Wonderland* [1] and a number of other children's books. He was also a pioneering photographer and one of the first to use photography as an art form. Most of his income came from his lectureship at Oxford University, UK. As a mathematician he wrote books on Euclidean geometry, symbolic logic and voting systems. 2015 is the 150th anniversary of the publication of Alice in Wonderland. Lewis Carroll in Numberland gives a comprehensive account of his work as a mathematician and a more general look at his life [2] [3].

The Stomachion

The Stomachion, which can be seen in Figure 1, is a very old puzzle similar to the Tangram. It was studied by Archimedes around 300 BCE. This, and other ancient works, were discovered in a prayer book, the original geometry and writings having been scratched off and overwritten with prayers. The Stomachion has been studied extensively. Many web sites give details, and a good summary can be found at [4].

Bill Cutler showed that there are 536 different ways of arranging the tiles to form a square. Three pairs of the tiles are always bound together reducing the problem to one with 11 tiles [6]. I have used the 11 tile version as a basis for this paper [6].

One English translation of the text referred to the Stomachion as a *parallelogram*. The Roman Poet, Decimus Magnus Ausonius refers to some tiles being equilateral triangles which maybe implies a parallelogram. For these reasons, as well as natural curiosity, I treat the general shape as a *parallelogram* of which rectangles are a subset. Strange things can happen in *Wonderland* especially if logic is applied!

Symmetrical and Non Symmetrical Designs

Each Stomachion can be defined by its three main angles and the length of its diagonal. I chose to look at examples where the three angles are in the ratio a:b:c where a, b and c are integers, see Figure 1.

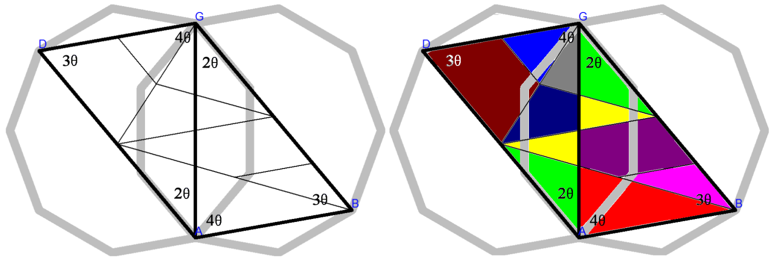


Figure 1 : *A Stomachion with angles in the ratio 2:3:4.*

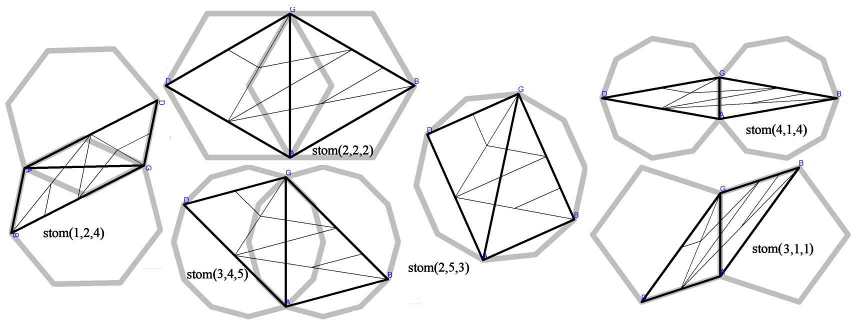


Figure 2 : *Fitting the stomachion to the vertices of overlapping regular polygons*

I am interested in exploring the relationship between these Stomachion ratios, the symmetries that can be developed and their relationship to regular polygons, see figure 2.

What is interesting, from a self-similarity point of view, is that a larger version of each primary tile can be produced from the primary tiles. As a consequence, any design or shape created using the primary shapes can be enlarged. In addition, the resulting enlargement can be enlarged again using the same process. I have, previously, called this property *preciousness* [7]. Figure 4 shows a parallelogram dissected into its primary shapes. It also shows how larger, similar shapes can be created. Figure 3 shows an ad hoc design of a puppy from *Alice in Wonderland* created from these tiles, and the start of a sequence of enlarged shapes using the scheme in figure 4. It is quite easy to create a design with one or more lines of symmetry. Figure 6 shows some examples. A Stomachion with angles in the ratio a:b:c will always fit into two overlapping regular polygons with n sides where $n = a + b + c$ [7]. I was expecting some connection between n and the potential for *rotational symmetry*. Figure 2 shows how the Stomachion will fit into overlapping regular polygons for a range of ratios. There are circumstances where the two polygons touch or completely overlap [7]. The ratios could be an arithmetic or geometric progression or something more sentimental like a wedding anniversary see Figure 7.

If a,b or c is a factor of 2n, then the triangle will fit an exact number of times around a point and display rotational symmetry. All of these designs can be expanded using the precious properties. They might take the form of a regular polygon, a polygonal border, a star or a saw tooth. See figure 7. Figure 5 is a matrix that shows the relationship between each of the primary shapes and its enlarged version. For example, row 3 shows us that the shape 3 needs 1 of each shape 3, 4 and 7 to create an enlarged version.

By repeatedly multiplying the matrix by itself it is possible to show that after a number of generations each primary shape will be included in any design and the proportion of each shape tends towards a fixed amount both by number and area. For example, shape 3 will tend to 3/16ths of the total number of shapes or 1/12ths of the total area. These are necessary conditions for the set of tiles to be called *precious* [7]. These matrices are also needed to calculate the *fractal dimension*. The mathematics is similar to the analysis of the connectivity of a network.

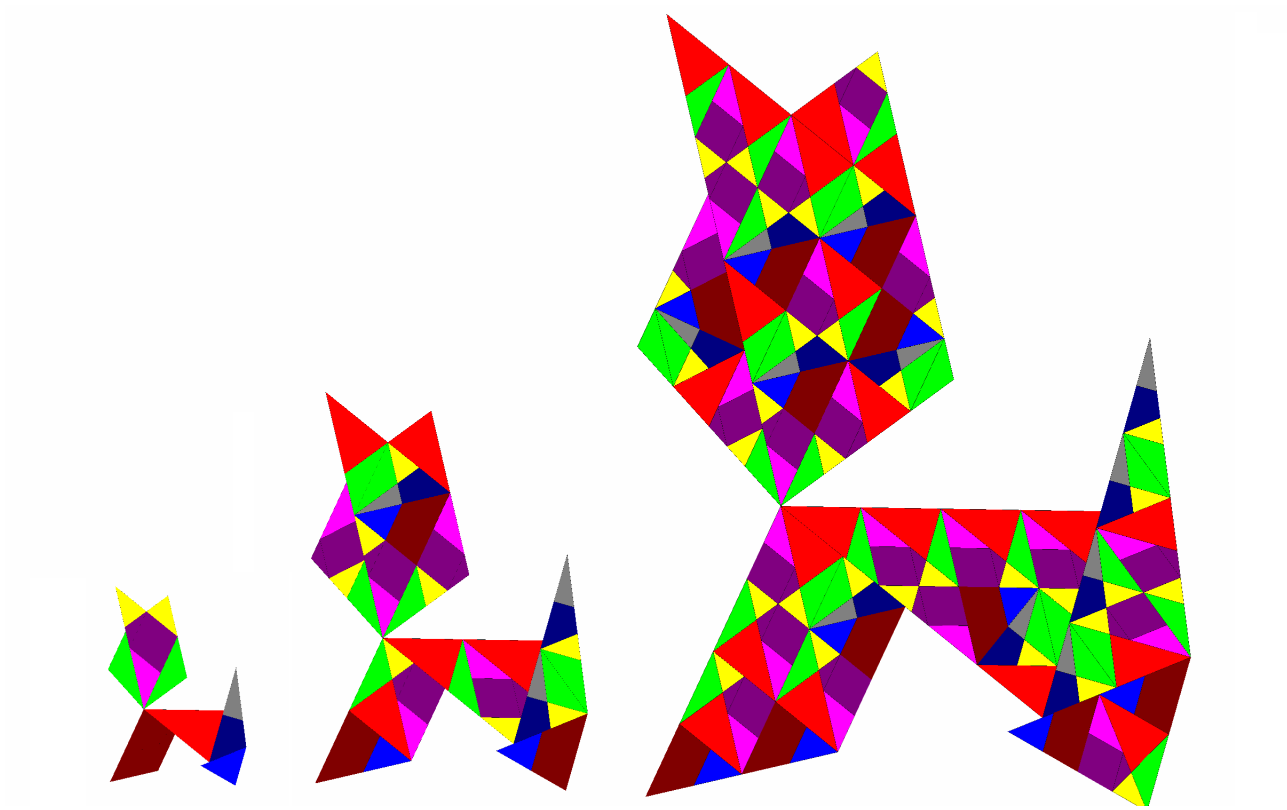


Figure 3 : *The assembly of an enlarged ad hoc design, showing the doubling of the size at each stage. The example shown is a puppy from Alice in Wonderland.*

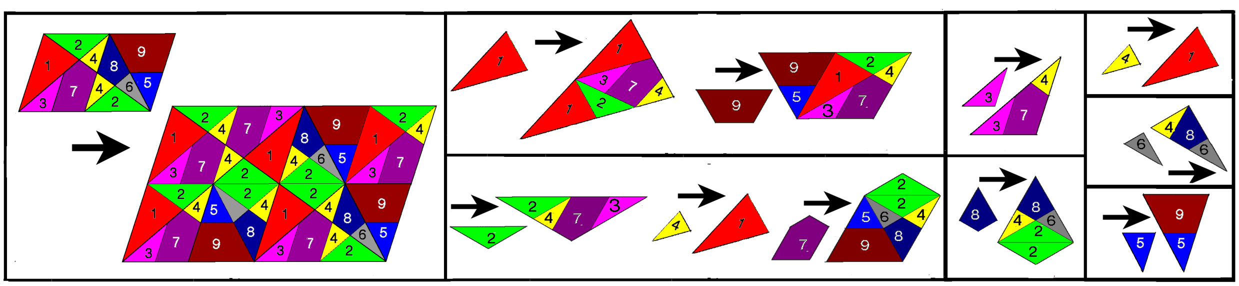
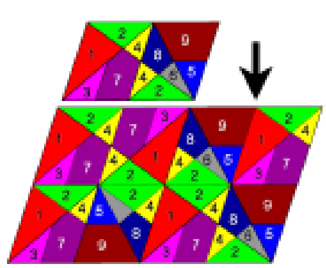


Figure 4 : *The dissection of the Stomachion into its Primary Shapes with subsequent enlargements.*



$$P = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 5 : *The Matrix showing the relationship between each Primary shape and its enlarged version.*

Creating fractals

By expanding a design we can create a series of two dimensional tiling designs that eventually fills the plane. If we expand the design without expanding one of the shapes then a fractal will be developed, as seen in figure 6. An important statistic is the *fractal dimension*. This will be 2 for a tiling in the plane and 1 for a line. The method for expanding a design, such as the Sierpinski sieve, counts the tiles and is not suitable for schemes with two or more tiles of different shapes and sizes. I have developed a different form of the usual equation. I have tested it on the Sierpinski sieve and in either case the same answers are obtained for the single tile fractal. For multi tile situations I use

$$D = \frac{\log(AP^2)}{\log(P)} \tag{1}$$

where P is the precious ratio and A is the fraction of the area converted. In the case of Sierpinski $A = 0.75$ and $P = 2$ which leads to the usual result. I often fill the spaces in a fractal, with a picture. In this work I have created and used a number of pictures from Lewis Carroll's stories. Examples relating to fractal dimension and a more detailed explanation can be found at [7].

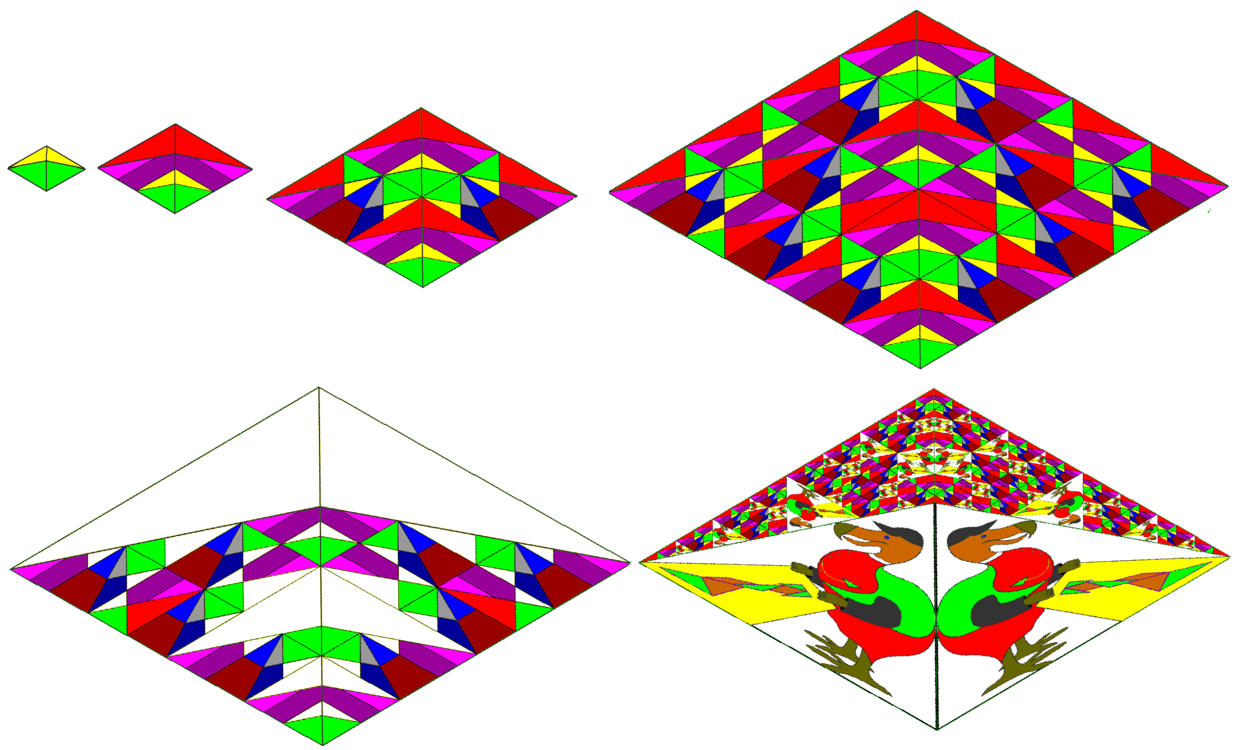


Figure 6 : *The development of a plane tiling, an undecorated fractal and a decorated fractal from the same base.*

The Pack of Cards from Wonderland.

To develop the pack of cards, I first designed the suits from the Stomachion shapes. Fractals were created, and the spaces filled with Lewis Carrol characters. Figure 8 shows the heart's design that features the Jack of Hearts stealing the tarts. The club's design features the dodo, which was Dodgson's nickname referring his stammer. Further examples of the pack of cards can be found at [7].

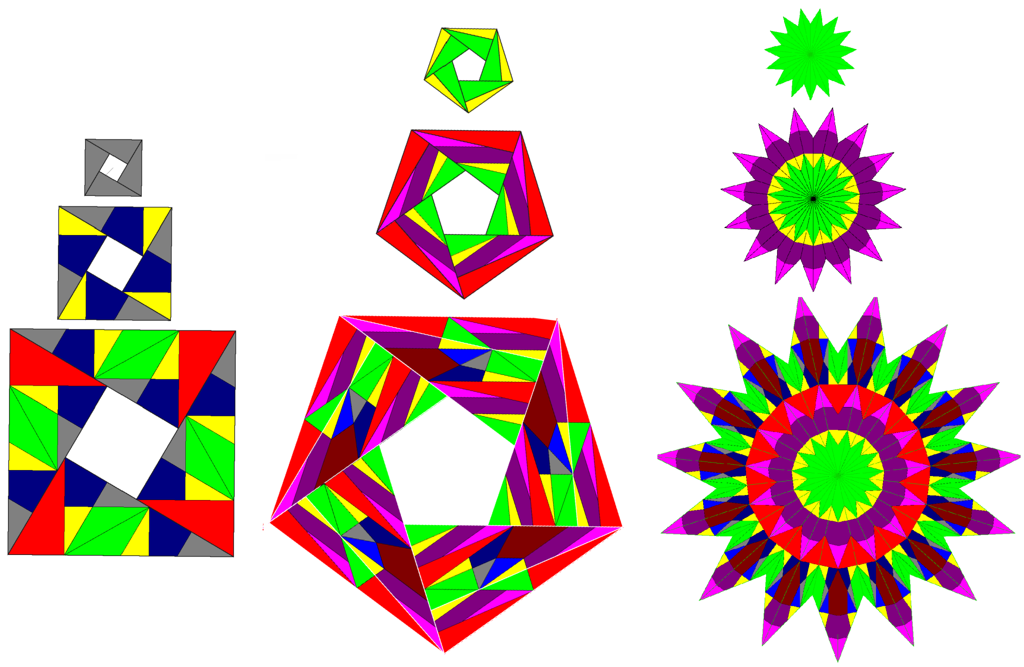


Figure 7 : *Examples of Polygonal Borders and a Star using angles in the angle ratios 1:2:2, 1:1:2 and 13:2:15 which is my golden wedding anniversary.*

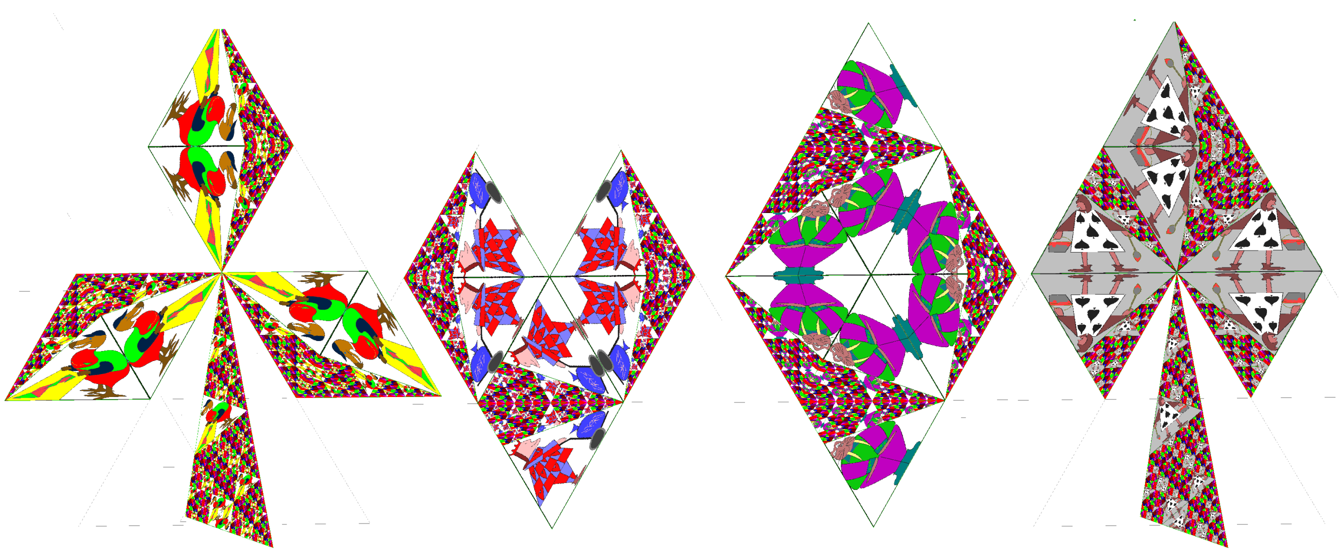


Figure 8 : *The development of fractals for each of the suits in a pack of cards.*

Creating a Chess Set

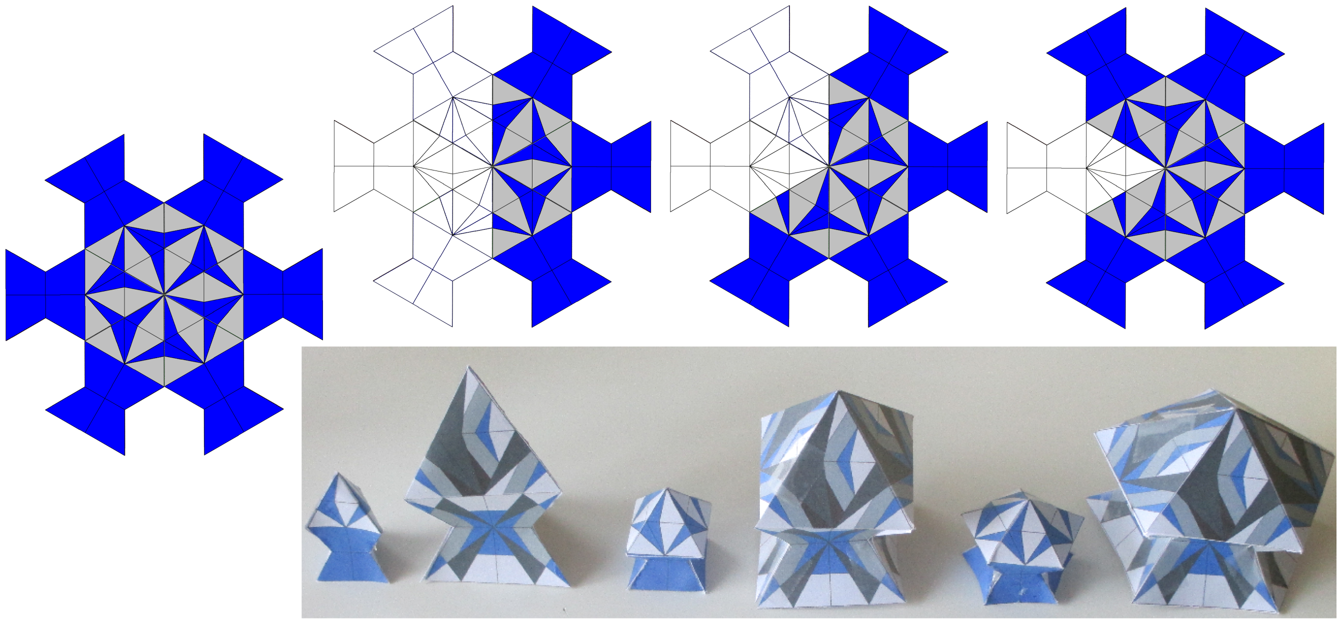


Figure 9: *The solid pawn based on a triangle, square and pentagon. In each case the next generation is also shown.*

In this section, we show how we can use the shape seen in figure 9 to make three dimensional chess pieces. This particular shape will fit six times around a point. We can make 3 and only 3 different solid shapes using this basic idea. This is a similar argument for the 3 platonic solids [8] using the equilateral triangle. These 3 nets will fold up to create chess pawns based on a triangle, square or pentagon, see figure 9. Other Chess pieces can be found at [7].

References

[1] Carroll L. Alice's Adventures in Wonderland / Through the Looking Glass.ISBN978-0-141-10068-9

[2] Wilson R. Lewis Carroll in Numberland. ISBN 978-0-141-01610-8

[3] Carroll L. Lewis Carroll's Bedside Book.Before ISBN British Library

[4] http://www.math.cornell.edu/%7Emec/GeometricDissections/1.2%20Archimedes%20Stomachion.html (as of Nov. 15, 2014)

[5] http://4umi.com/play/stomachion/

[6] http://www.math.ucsd.edu/%7Efan/stomach/tour/stomach.html (as of Nov. 15, 2014)

[7] http://www.grandadstan.com/sjspdf/Baltimoreslides2015.pdf (as of Nov. 15, 2014)

[8] http://meandering-through-mathematics.blogspot.co.uk/2011/11/why-are-there-exactly-five-platonic.html (as of Nov. 15, 2014)