Magnetic Circle Packing in Creative Outreach and Refreshment

Peanuts, pennies and "patanons" and the intriguing "iron ratio" 0.701….

Robin Linhope Willson
Professor Emeritus, Brunel University, London, England
and Ciencias y Artes Patagonia,
Puerto Madryn, 9120 Argentina

Abstract

Since 1993 the Magic Penny Trust supported by Brunel University, the Royal Institution and the Institute of Physics in London, have been developing novel science outreach programs using magnets and plated steel coins. Interactive experiments have been designed related not only to magnetism and physics but also to mathematics. Here I describe the background to the recent uncovering by Ciencias y Artes Patagonia of the interesting packing properties of a particular pentagon-based circle array, the *Patanon.* This uncovering has raised intriguing questions as to why such pentagon-based circle arrays seem to have received so little attention and why the regular pentagonal close packed inscribed circle relative diameter constant: 0.701…, the "*iron ratio*", is hardly known.

Figure 1. *"Science on Peanuts: Greetings from a wintry South America in starry-skied Patagonia."*
- a hexagonal snowflake-like star array of 13 British 5 penny (5p) coins balanced on their rims and held together by the magnetic field from ceramic block magnets out of the frame above the coins.
(A sister image entitled "Science on a Shoe String" was photographed around the same time. See [2].)

The unedited photograph (Figure 1) was taken for a winter greetings card for an astronomer friend. He was of sufficient age to appreciate that 1/6 net, on the paperback booklet, refers to one shilling and six pence in the old British pre-decimalization copper and cupro-nickel, non-magnetic, coinage. Today, several UK coins are made of copper- or nickel-plated steel and as such are magnetic [1].

Most American readers, particularly from Baltimore, the birthplace of the Star-Spangled Banner, will quickly appreciate that although the shape of most of the stars on the book-cover (Figure 1) match the hexagonal coin array, only one is the same shape as the stars on the national flag. The stars on the US flag are pentagonal in shape with five-fold symmetry. They are more like the coin array in Figure 2, formed from the same 5p coins as previously but with a central coin surrounded by 5 close packed larger 2 penny (2p) coins.

The ratio of the diameters of the UK 2p and 5p coins, 25.9 mm (1.02”) and 18 mm (0.71”) respectively, is equal to 0.694. This is within one percent of the ideal ratio for pentagonal circle close packing, the “iron ratio” 0.701… derived using simple geometry (see later).

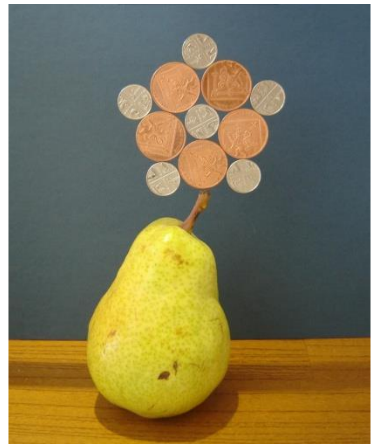


Figure 2. *An array of real UK 2p and 5p coins, balanced on the stalk of a pear. No glues or photoshop!*

Of course to most casual observers such photographs are put down to trickery, hidden glues or tape or being simply “photoshopped”. Fortunately, others are less skeptical and are truly amazed. “Why didn´t we do this at school?” “Much more interesting than the stuff we did with bar magnets, iron filings and paper clips.” Indeed! Suitable inexpensive ceramic magnets have been available for over 60 years.

US coins are not magnetic (except for those made in 1943 - the war-time “wheat pennies” see Figure 3.) However, steel washers, nails and paper clips, can be easily made to undergo similar amazing balancing feats with readily available ceramic magnets, that have been suitably protected from chipping. See [1].

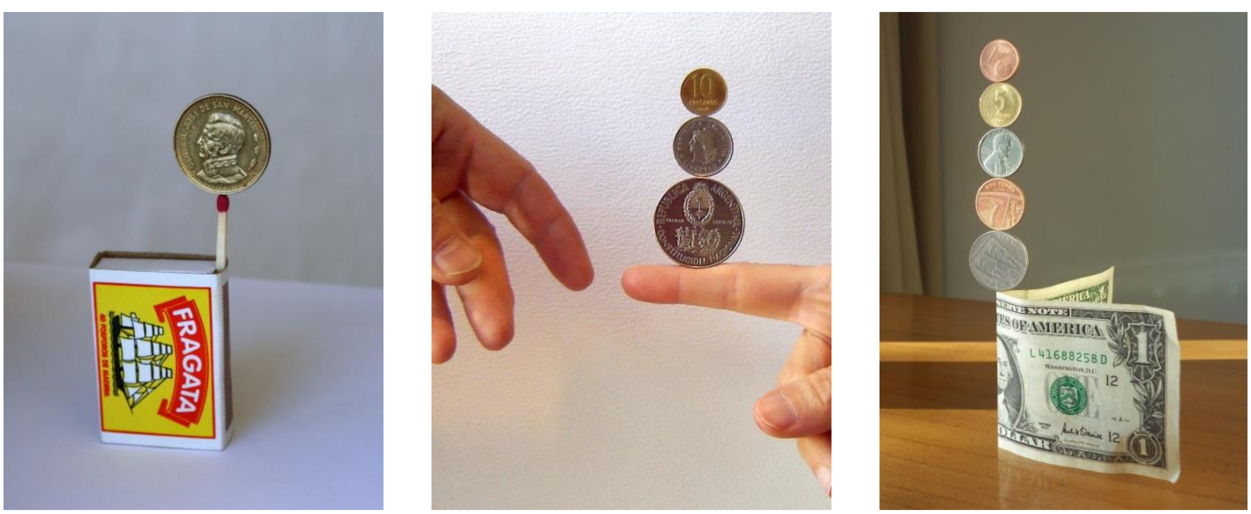


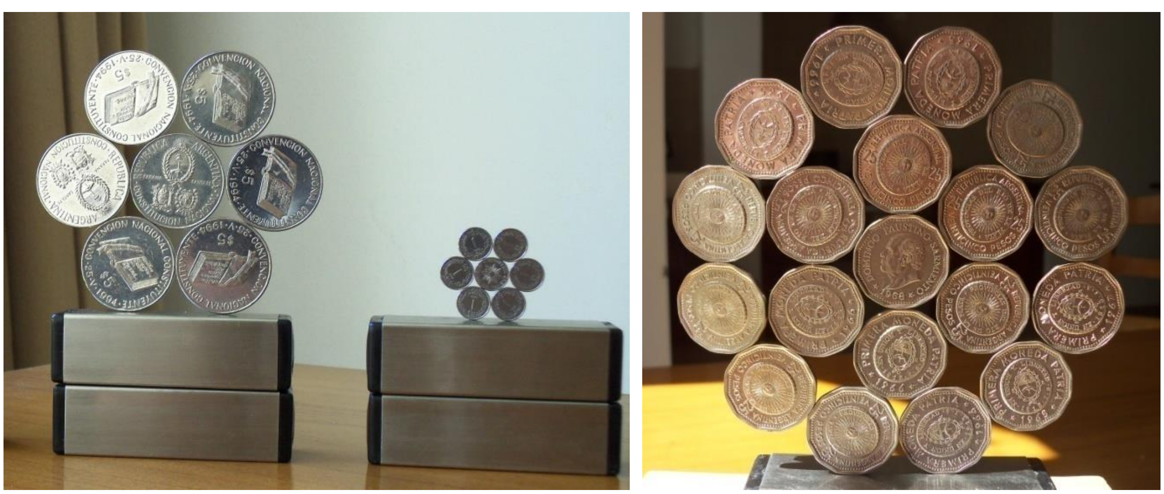
Figure 3. *Balanced Coins of Argentina (left and centre) and EC, Argentina, US and UK (right).*

Figure 4. *Magnet arrangement for "Balancing 4 Coins on a Wine Glass" and "Science on Peanuts".*

Ciencias y Artes Patagonia has now identified over 600 different magnetic coins from 70 countries. They range in diameter from the 1981 Singapore ten dollar (40.5 mm) and 1994 Argentina five pesos (35.0 mm) to the tiny 1989 Uruguay 1 peso (12.0 mm) shown below. Photographs and videos of both static and rotating arrays have been published [2]. Here I will focus on hexagonal and pentagonal arrays constructed using ceramic block magnets covered in non-magnetic stainless steel like those used above. Occasionally, the magnetic field has been enhanced by attaching a rare earth, "neo", magnet to the block. Later I will describe studies with magnetic mats similar to those used in teaching and business displays.

Hexagonal- based standing arrays

The fact that six circles of equal radius will fit exactly around a circle of the same radius is common knowledge.



.

Figure 5. *Hexagonal arrays of coins of Argentina and Uruguay balanced upright on their rims due to the magnet field from the stainless-steel covered magnets on which they are standing. The coins on the right are actually 12-sided rather than circular allowing them to be even more closely packed.*

Figure 6. *Standing hexagonal arrays illustrating the power of magnetic fields and the mechanical triangle-like rigidity resulting from three circle contact.*

Because of their high packing density large solid hexagonal arrays are easy to construct. Photographs of an array of 168 old German 1 pfennig coins suspended from a very strong (but potentially dangerous) rare-earth magnet have been published [2].

Pentagonal-based standing and hanging arrays

Figure 7. *A two-coin pentagonal array of coins of Argentina and Uruguay on (left) and a two-coin circular array UK 2p and 5p coins. In both cases the ceramic magnets are reinforced with "neo" rare earth magnets.*

Figure 8. *"Northumbrian Rose" of UK coins (1p, 2p, 5p and 10p totaling exactly one UK pound) and "Rose of Patagonia" containing current and old coins of Argentina. Both arrays can be balanced standing or suspended, using only the safe ceramic block magnets placed above and below the coins.*

Planning and Investigating Arrays using Magnetic Mats

Just as large jigsaws would be difficult to complete if it was not for the tabs and slots keeping the pieces together, so investigating close packing with coins is difficult unless the coins can be temporarily kept in place. Magnetic mats and magnetic coins overcome this problem. The mats can be laid horizontally or attached to a board and propped vertically.

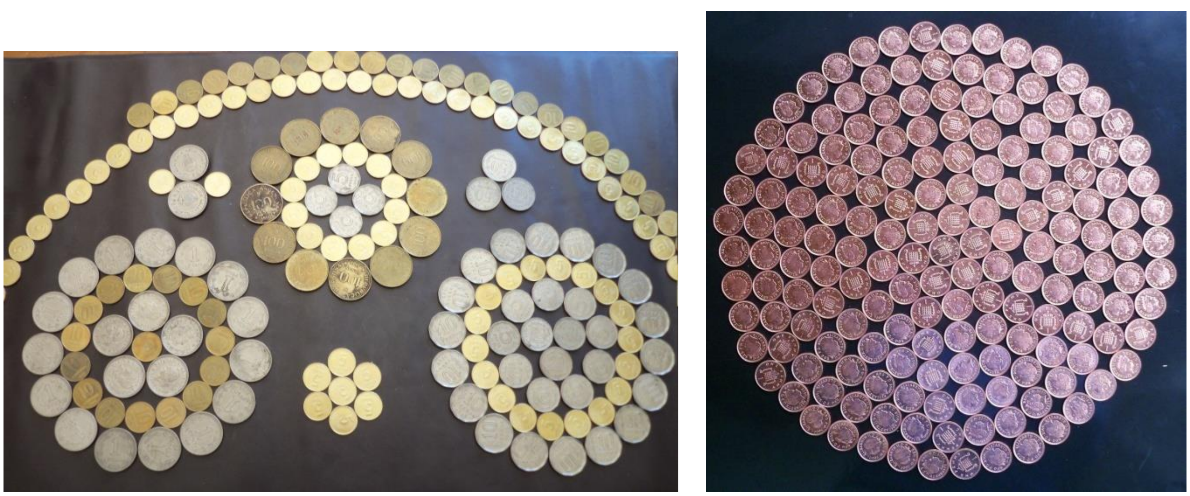


Figure 9. *Experimenting with two-coin curves and circular arrays with different centres and packing styles using magnetic mats and differently sized coins.*

Figure 10. left: *a 1-5-10-20-20 rose of coins of Brazil and UK (inner ring) surrounding a 1 cent coin of the European Community.* **right:** *a large two coin pentagonal array in which many of the smaller 5p UK coins are old and non-magnetic but are held in place by being gripped by the magnetic 2p UK coins surrounding them.*

Uncovering of the tesselating pentagon-based "Patanon"

F**igure 11**. *Greeting card coin designs based on a central decagon, inset with a pentagonal array.*
left: *of 2p and 5p UK coins with some of the 2p coins covered in yellow or red adhesive plastic to emphasise some of the decagon arrays. The outer double circle is of 5 and 10 centavo coins of Argentina.*
right: *of 5 centavo coins of Argentina and steel washers of similar size to the 1 peso coin of Uruguay used in Figure 7. The outer double circle of 5 and 10 centavo coins of Argentina is incomplete due to lack of 5 centavo coins.*

The study of circle packing has a long history going back to Descartes, Kepler and beyond. Recently, there has been a strong resurgence of interest due principally to the advent of computers. Many studies have been done concerning how different numbers of circles or discs of one size, or how efficiently any number of circles of any size, will pack into a triangle, square, or circle. But what seems to have received little attention is as to how circles of just two sizes can fill a space.

My considerable recent interest began when contemplating how to design an image I might put on a Christmas greetings card last December. It was then that for the first time I realized that the arrays in Figure 12 might be possible.

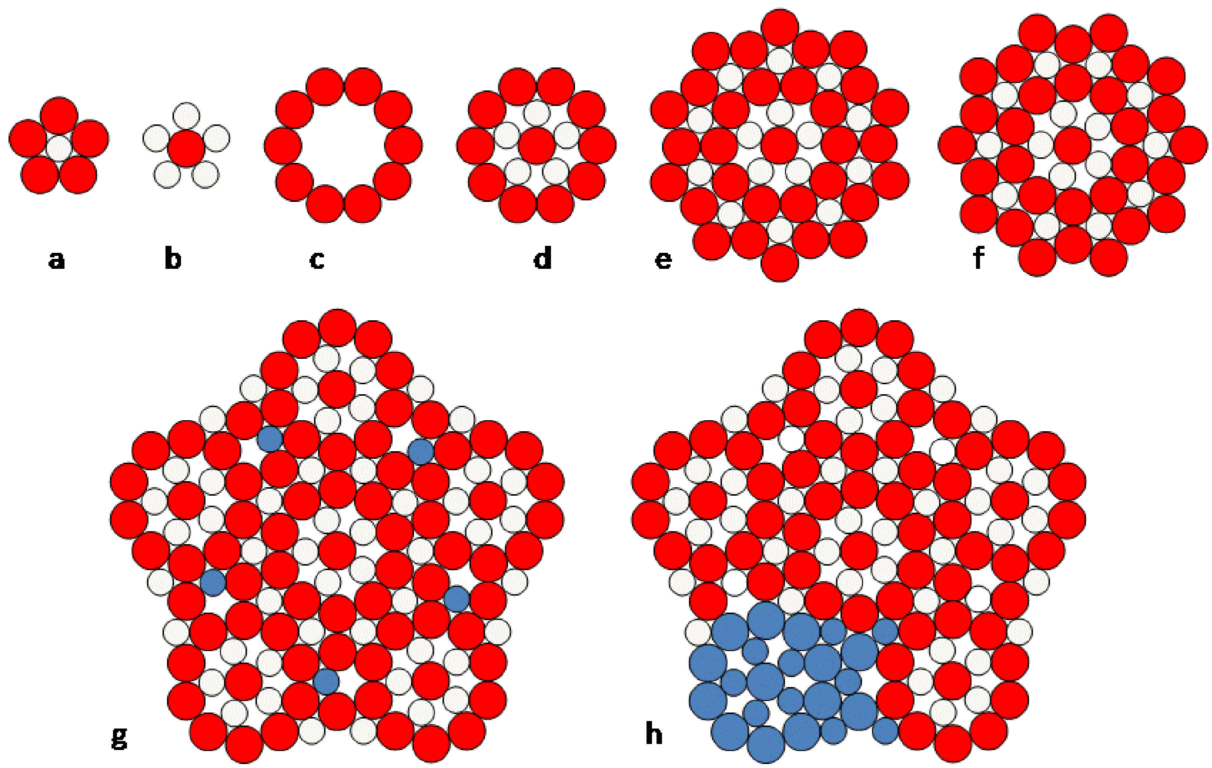


Figure 12. *If a small coin fits almost exactly into the centre of a ring of larger coins of the same diameter* **(a)** , *then a pentagram-shaped star* **(b)**, *made up of five of the smaller coins surrounding, one larger coin will fit almost exactly into a decagon-shaped circle of ten of the larger coins.* **(c-d)**
The decagon-centred array can be extended symmetrically in all directions by adding further coins such that additional decagons radiating from the centre are formed **(e-g).** *A beetle-like array occurs as a repeating unit (shown in blue/ light grey) circling the central decagon* **(h).**

The symmetry of the beetle-like unit is such that successive pentagonal-like circles of the units can be packed together with the side of each "circle" increasing by one beetle unit.

During the subsequent construction of the festive arrays (Figure 11), I began to ask why I had not read about this before. Furthermore it was clear that as more and more beetle units were added, the process could be extended indefinitely and that soon vast areas of parallel-packed beetle units could be formed which did not include the original decagon. The possibility arose, that here might be a basis for forming a unit that could tessellate a surface. By tracing around the beetle unit as it lay packed in the overall array and including some of the inter-circle spaces around it, a symmetrical curved beetle-like shape might emerge. Its curved edges would be made up of interconnecting convex and concave arcs of radii equal to the smaller and larger coins. Thus the "Patanon" was born. (Figure 13)

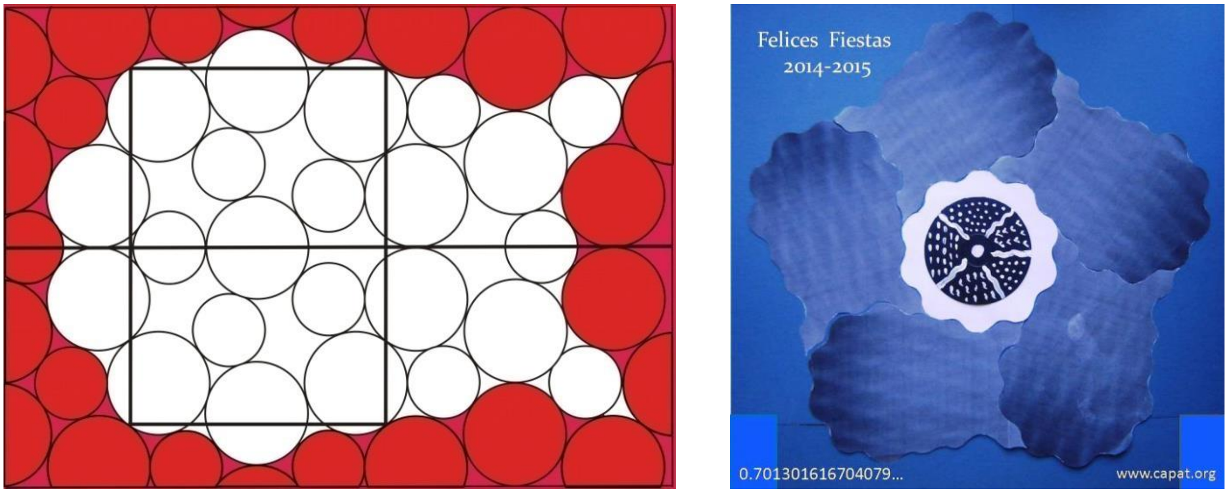


Figure 13. *The "Patanon" based on discs of two distinct diameters related by the iron ratio 0.701…..* **left:** *showing the horizontal line of symmetry and the rectangle of symmetrical folds (see text)* **right:** *a festive card showing paper "Patanons" packed around a central decorated "curved decagon".*

Of course the *Patanon* need not only be based on coins. The overall shape can be constructed from any two sized circular objects, even DVDs and coffee jar lids, provided the ratio of their diameters is equal to the "iron ratio" approximately 0.701. (Put simply the smaller object must be of diameter approximately 70 percent of the larger.) *Patanons* can be hand drawn and cut, or laser cut from paper or other material and decorated to choice. In all instances the length and width of the overall *Patanon* is directly, and only, dependent on the diameter of the larger object and the *iron ratio*.

As a mathematical teaching or outreach aid, *Patanons* are rich in creative opportunities, particularly when their underlying circles and their centres can be seen. The associated similar pentagon-related triangles, parallel lines and lines of partial symmetry are full of mathematical challenges. A paper *Patanon* can be folded inwards along the sides of a rectangle whose corners lie on the points of tangential coin contact shown. The folded parts fit exactly together in agreement with the area of the *Patanon* being twice the area of the folding rectangle shown above. To golden ratio (phi) enthusiasts, the occurrence of numerous inscribed golden rectangles will also not be surprising with pentagons being so much involved. But ask not how the iron ratio is related to the golden ratio. Consider how the golden ratio relates to the iron. After all, round pebbles and fruit existed long before classical geometers. If the *iron ratio* is represented by the classical symbol for iron, but here simply as FeR, then it can be easily shown that:

$$\mathrm{FeR} = (1-\sin(36°))/\sin(36°) = 0.701301616704079\ldots\ldots$$

$$\text{and} \quad \mathrm{phi} = (1 + \mathrm{FeR})\cos(18°)$$

But again, as with our magnetic uncovering, "Why have pentagon-circle connections and such equations been so much overlooked?" In presenting this paper I hope to stimulate discussion and wider enquiry. Thank you to all those connected with the Magic Penny Trust and Ciencias y Artes Patagonia who have provided such invaluable, enthusiastic and open-minded, support. See [1] [2].

References

[1] R.L. Willson, P.A. Riley and D.J. Harris, *Investigating Magnetism- the Science and Art of Magnetic Coin Tricks, 4th edn. 2014,* Brunel University, London ISBN: 1-872166-38-5 1995.
[2] www.magicpenny.org www.capat.org www.MagneticCoins.info and YouTube: MagneticCoins