

The Geometric Studies of Some Mosaic Design Compositions and Puzzles Presented in a Thirteenth Century Treatise

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Abstract

Interlocks of Similar or Complementary Figures is the only treatise about girih construction that has survived for many centuries. It is believed that the document was written sometime between the 13th and 15th centuries by an anonymous mathematician/craftsman. This is the only document remaining from the medieval Persian that demonstrates how a girih can be constructed using compass and straightedge. The treatise includes some puzzles for polygon transformations. This article includes the study of many puzzles and girih patterns that have been documented in the treatise. All images, except for photographs from the treatise, have been created by the author using the Geometer's Sketchpad (GSP) software program.

1. Introduction

In addition to decorated tiling on the surfaces of existing structures from the past, there are a few documents about Persian mosaic designs that have survived throughout many centuries and have reached us today. These documents, as scrolls and treatises, include many intriguing girih patterns, the fundamental regions for tiling of ornamental designs, as well as some compass-straightedge constructions and puzzles used for transforming one geometric shape to another.

On Those Parts of Geometry Needed by Craftsmen is a treatise written by the well-known 10th century Persian mathematician Abûl-Wefâ Buzjâni, who lived and worked in Baghdad, perhaps at the House of Wisdom (Bayt al-Hikma), which was a major intellectual center during that era. The treatise does not include any ornamental designs. However, it presents a number of compass-straightedge constructions, which are used in the composition of such patterns.

The treatise by Buzjâni was originally written in Arabic, the academic language of that era in the East and was translated into Persian during several periods. In one of the translations by the 15th century Persian mathematician Abûl-Es-hâgh Koobnâni, another treatise in its Persian original, was appended: *Interlocks of Similar or Complementary Figures* [1]. Some historians believe that the author of this document is a 13th century anonymous mathematician/craftsman [5]. However, in a recent Persian book that includes modern translations of both Abûl-Wefâ's treatise and this document, the author has been identified as Koobnâni [2]. The document itself was not signed by Koobnâni and was not dated. Moreover, it is not clear if the two treatises were together originally. Therefore attributing the document to Koobnâni is only speculation. It is essential to note that the treatise *Interlocks* is the only known practical manual from the past that provides "how-to" instructions for drawing two-dimensional girih patterns. "The treatise shows that the girih mode was conceived as a system of proportionally related geometric patterns harmoniously interlocking with one another" [4].

The aim of this article is to study and analyze a selected set of girih patterns and puzzles that have been recorded in the *Interlocks* treatise. The selection includes all those interesting girih patterns and puzzles that form the main body of the treatise.

2. Interlocks Treatise's Puzzles

This section is devoted to the study of some puzzles in the treatise that are used for transforming one geometric shape to another.

2.1. Constructing a Kokabi Star. The treatise began with an interesting but simple puzzle (the left image in Figure 1). The second image is the construction of the puzzle using GSP. By cutting two decagons and pasting pieces together as in the first and second images, one is able to construct a larger decagon with a star at the center, with the shape of the great pentagram (*Kokabi* star in Persian), which is different from the pentagram $5/2$ star polygon. *Kokabi* can be constructed using the segments that constitute the $10/3$ star polygon (third image in Figure 1). It is interesting to note that the outline of the two Penrose quasi-crystal tiles, *kit* and *dart*, constitute a petal of this star as in the third image. The right image in Figure 1 is a *Kokabi* tiling on the surface of a ceiling at *Kaseh Garan* School, Isfahan, Iran, that was studied in an article by the author [6].

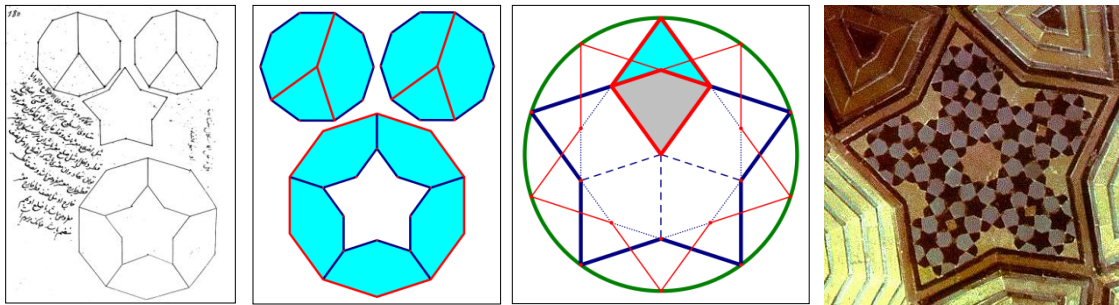


Figure 1: *Constructing the Kokabi star and its application to Persian architecture*

2.2. Octagram and Square Transformations. The left image in Figure 2 presents a puzzle that transforms a square to an octagram. The right image is a GSP construction of the puzzle, where the construction has followed the lines of cuts suggested by the treatise. It seems the treatise suggests that the square should be cut along with the lines that constitute four congruent right triangles with the small sides congruent to one half of the longer side, as is exhibited in the right image of Figure 2.

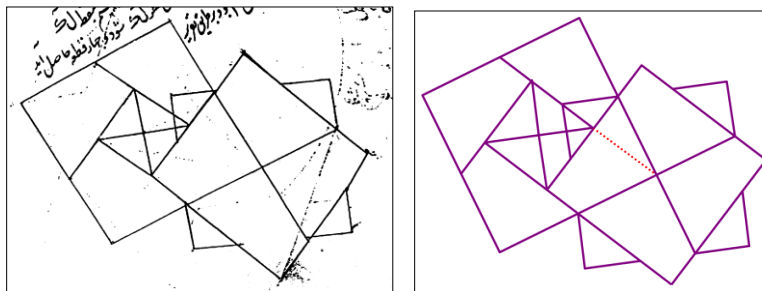


Figure 2: *Transforming a square into an octagram and vice versa*

Even though such a cut creates an octagram as in the right image in Figure 3, this octagram is not equilateral, which is a condition that seems necessary in puzzles like this.

For cutting a square into pieces to create another square with lines for making an equilateral octagram, one may do as follows: Suppose AB is a side of a given square. Drop a perpendicular

to line AB at B and call it line x . On x make an arbitrary point C . Find point D in such a way that $BC \perp DC$ and $BC = DC$. Find point E in a way that $BC = CE$. Now find point F with the condition $ED = EF$. Connect A and F and make a segment parallel to AF from C to cut AB at G . G is the desired point. This process is an easy task using two circles that have been constructed in the left image of Figure 4. Now if the square is cut according to segments AG and GB , as in the second image in Figure 4, and a small square with sides $AG - BG$ is added one can make a bigger square that can be transformed to an equilateral octagram (third and fourth images in Figure 4).

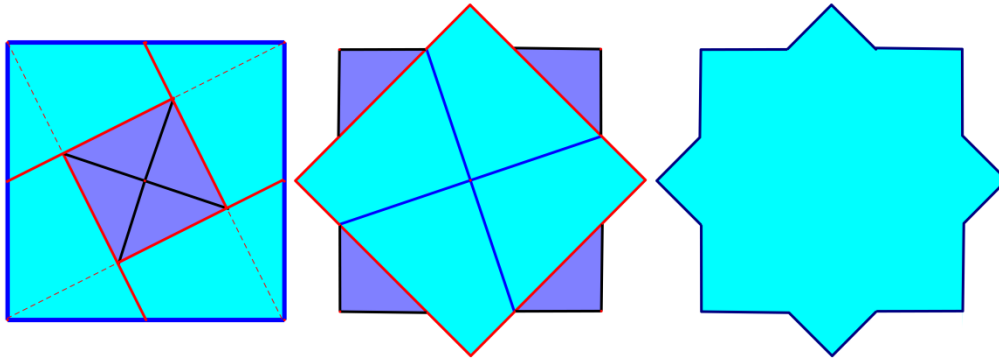


Figure 3: Transforming a square to an octagram, which is not equilateral

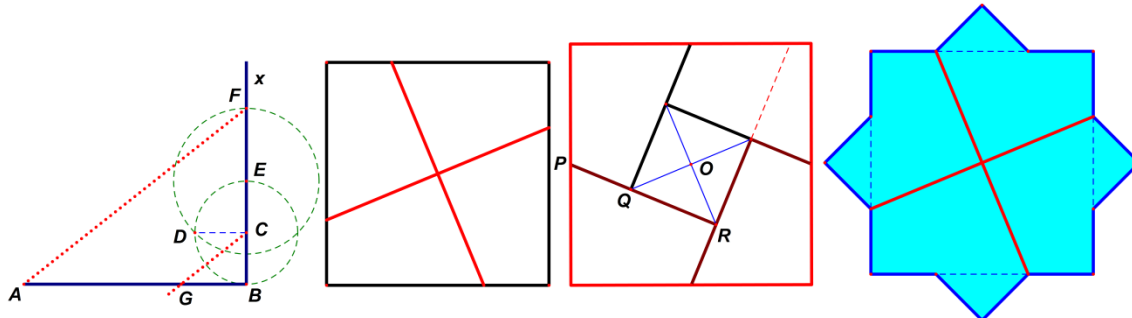


Figure 4: Transforming a square to an octagram, equilateral case

Sketch of proof: Segment AB and arbitrary point C on the perpendicular line to AB , x , are given. Find points D , E , F , and G according the above instruction. Suppose $m(BC) = a$. Then $m(DE) = \sqrt{2} a$. As a result $m(CF) = (\sqrt{2} + 1)a$. Note that CG is parallel to AF . So ΔBGC and ΔBAF are similar. Therefore, there exists a number b such that $m(BG) = b$ and $m(AG) = (\sqrt{2} + 1)b$. The cuts in the second image in Figure 4 are based on b and $(\sqrt{2} + 1)b$. Therefore, the measurement of a side of the small square in the middle of the third image is $m(QR) = (\sqrt{2} + 1)b - b = \sqrt{2} b$ and therefore $m(PQ) = b$. But then employing the isosceles right triangle ΔOQR one can obtain that $m(QO) = b$. Hence $m(PQ) = m(QO)$ and as a result the octagram on the right image in Figure 4 is equilateral. \square

This transformation is easily reversible by cutting an equilateral octagram along the lines that connect vertices according to the right image in Figure 4. The job of introducing a compass-straightedge process for beginning with a square (as in the third image in Figure 4) and ending with the equilateral octagram (as in the right image in Figure 4) by some appropriate cuts seems more challenging and is left to interested readers.

2.3. Hexagram and Square Transformations. The image on the top left in Figure 5 is from the treatise. A regular hexagram, as in the top middle image in Figure 5, is given. Label a few of its vertices as in the image. Extend AE to AK in a way that $EK = EG$. Construct the semicircle AK to intersect the extension of GE at R . Find S on AK in such a way that $SE = RE$. Note that SE is congruent to a side of the desired square. This is a fact that can be proven using the relationship between the area of the hexagram and the measure of segment SE based on the equation $SE^2 = ER^2 = AE * EK = AE * EG = 3a \cdot 2h = 6ah = 12 (1/2 ah)$, where a is the measure of a side of the hexagram and h is the measure of the altitude of $\triangle CBD$, which is one of the 12 congruent equilateral triangles that constitute the hexagram. M is the midpoint of AS , and L is the midpoint of SE . Find N on HG in a way that $GN = AM$. Drop perpendicular lines from S and L to MN to find points Q and P . Note that $LP = LS = LE$. This is a condition that is imposed by the fact that each side of the assembled square should be congruent to segment SE . Therefore, $m(PL)$ (from $PLSQ$) + $m(PL)$ (from $PLEFGN$) = $m(SE)$. Make similar cuts to the other part of the hexagram. Now arrange pieces to compose the square in the top right image in Figure 5.

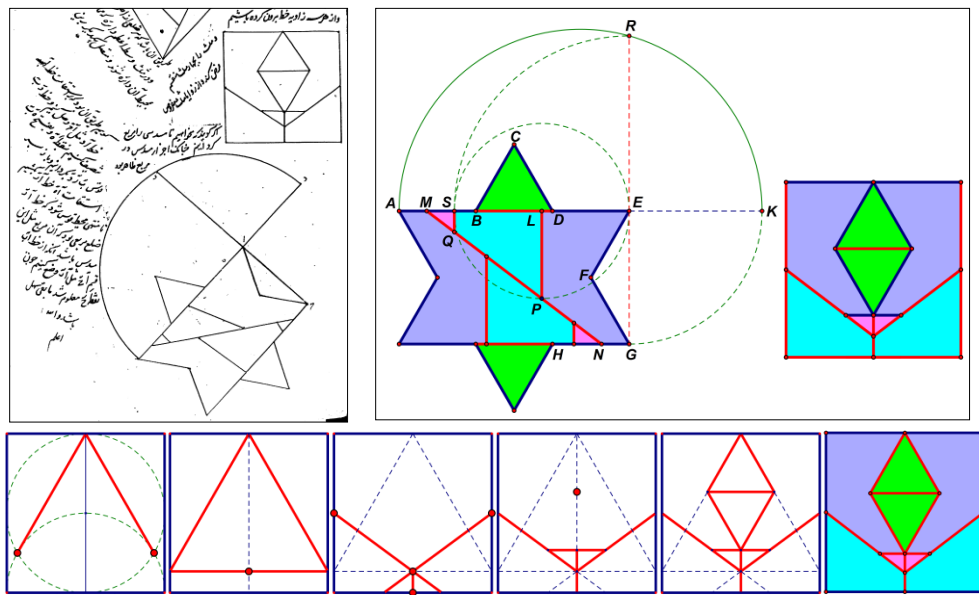


Figure 5: Transforming a hexagram into a square and vice versa

It is not difficult to confirm the validity of the transformation from square to hexagon as is presented on the bottom images in Figure 5. However, the treatise does not give any instruction in this regard. The series of images on the bottom part of Figure 6 presents such a process. At first a 60° angle can be constructed so that its vertex is on the midpoint of a side of the square and then using this angle one should construct an equilateral triangle so that the other two vertices are located on two sides of the square. Next, using the midpoints of the square's sides, two trapezoids are constructed according to the images. The rest of the decomposition can be followed easily throughout the last few images in Figure 5.

2.4. Regular Hexagon and Equilateral Triangle Transformation. A regular hexagon is given. Label it as the middle image in Figure 6. Draw the diagonals AC , CE , and EA . $\triangle ACE$ is an equilateral triangle. Find P , the midpoint of CD and connect it to A . Repeat this process for EF and AB to construct $\triangle LMN$ (which is an equilateral triangle). Point O , the center of the hexagon, is the centroid of $\triangle LMN$. The medians divide $\triangle LMN$ into 6 congruent right triangles as in Figure 6. Now by re-arranging the pieces that constitute the hexagon, an equilateral triangle will be created (Figure 6. Right).

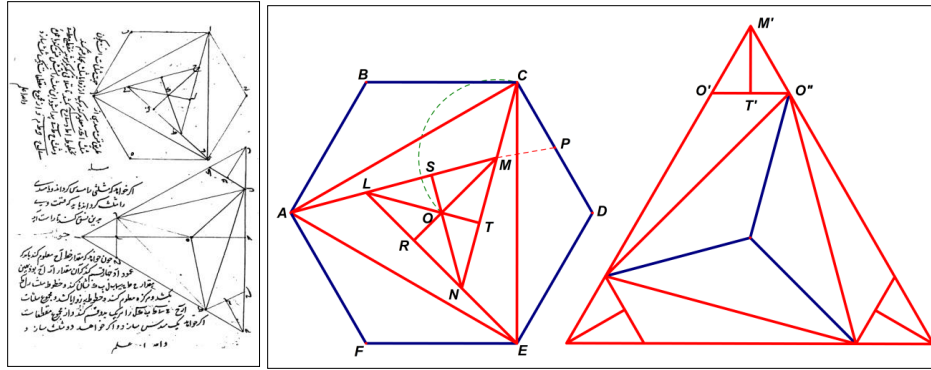


Figure 6: Transforming a hexagon into an equilateral triangle and vice versa

So far the mathematical proof to show the validity of this transformation has not been found. Moreover, the reverse process seems more challenging.

In order to show that this transformation is valid, one should prove $m(MC) = 2 m(OT)$. This is a necessary condition as is presented in the triangle on the right because we should have $m(O'M') = m(O'O'') = m(O'T') + m(T'O'')$. But since $\angle OMS = \pi/6$, $m(OM) = 2m(OT)$. So it is sufficient to prove that $m(MC) = m(MO)$. Using GSP one observes that a circle with center M and radius MO passes through C . But this is only an observation and should be proven mathematically.

3. Interlocks Treatise's Girihs

In this section a number of interesting girihs that are recorded in the treatise are presented and analyzed.

3.1. An Octagonal-Hexagonal Girih. The left image in Figure 7 is from the treatise. The square $ABCD$ is given as in the middle image in this figure. Pick arbitrary point P on diagonal AC . Find G on CD in such a way that $CP = CG$. Find H on BC so that $CH = PG$. Make equilateral triangle $\triangle CMH$. Find F on AC in such a way that $MF = CP$. Find K on BC in a way that $FH \parallel OK$. Find T on CM so that $CT = CK$. Using K and T , complete a quarter of a regular hexagon as in the figure. Compose the same quarter of the hexagon in a rotational fashion around the other vertices of the square to find Q, R , and S . Polygon $QRST$ is a square. Using this square one can construct a regular octagon as is presented in the middle figure. It is interesting to note that P and K are independent. Therefore, it does not change the end result if one changes the position of the arbitrary point P .

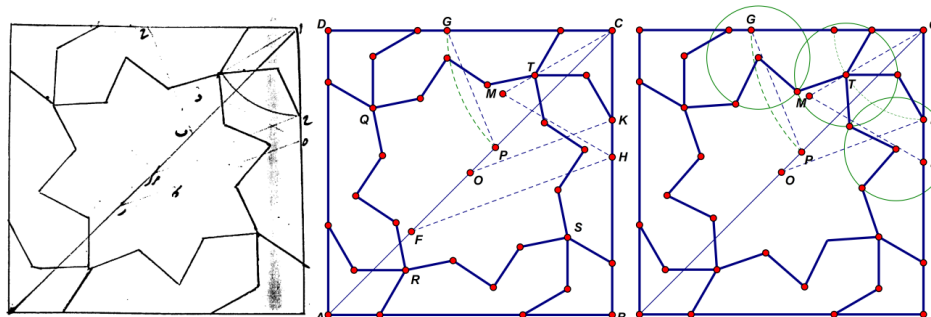


Figure 7: The left image is from the treatise, the middle image includes a regular octagon, the right image is a girih where all sides are congruent.

It is not difficult to realize that even though the sizes of the sides of the octagon and hexagon are very close to each other, they are not congruent. Using the four vertices Q, R, S , and T as the center of some

circles with radius congruent to the sides of the hexagon, one can compose the right girih, where all sides are congruent. In fact, from the left image in Figure 7, it seems that the congruency of the sides was a condition. Figure 8 illustrates the tessellations for both girih.

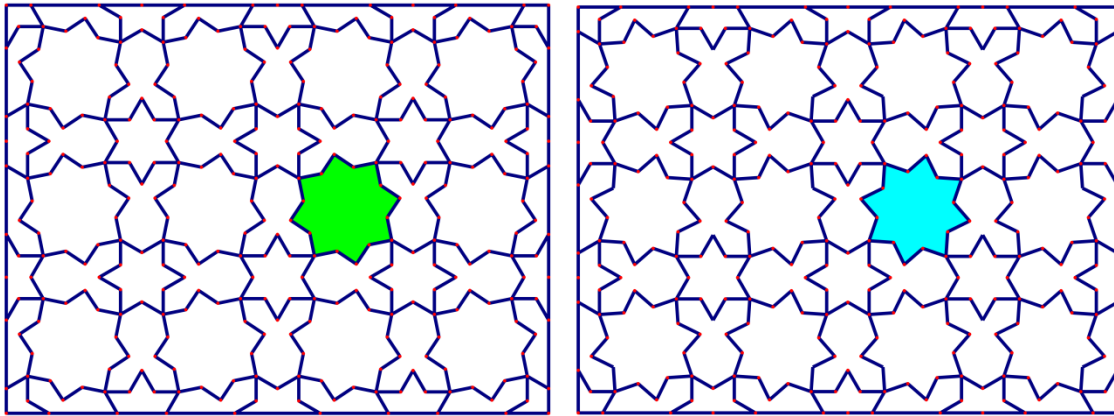


Figure 8: (Left) *The regular octagon-hexagon tiling;* (R) *A tiling with congruent sides.*

3.2. Making a Girih for a 10-Petal/12-Petal Combination Rose Tiling. The left image in Figure 9 is a sketch of a girih as appears in the *Interlocks* treatise. This girih creates a pattern that includes both ten and twelve petal roses. The second and third images illustrate the two girih of 10-petal and 12-petal roses separately. The last image shows how this combination occurs.

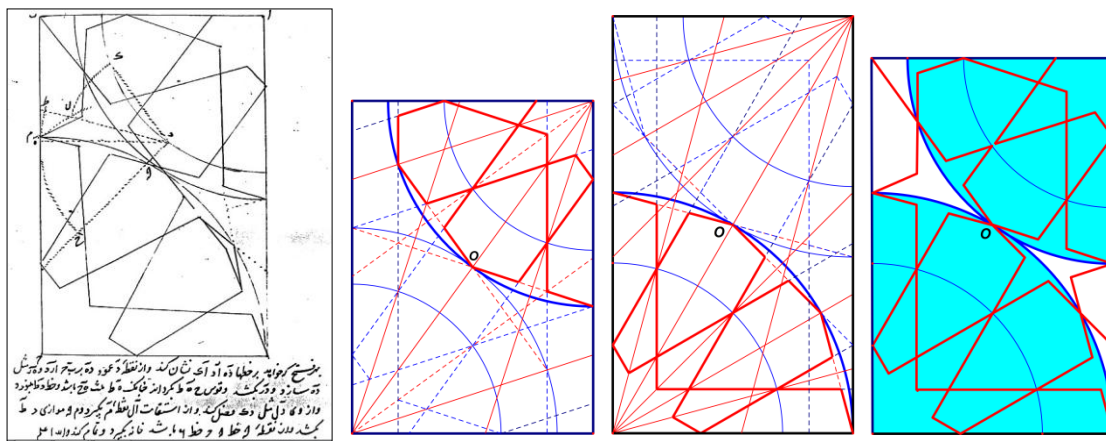


Figure 9: *A girih in the Interlock treatise, and the two girih that create this design*

The method for the creation of the two girih of 10-petal and 12-petal roses, the *Radial Grid Method*, is attributed to Maestro Maher-al-Naghsh. He used this method for the construction of a different girih, but one with sufficient expertise in girih construction can discover the required general steps for the composition of a large number of girih in different shapes [7].

The five volume book in Persian, *Construction and Execution of Design in Persian Mosaics*, authored and illustrated by M. Maher-al-Naghsh [3], is the most comprehensive recent book about the construction of the Persian mosaic designs. M. Maher-al-Naghsh, a professional artisan, who inherited his profession from his ancestors of several centuries and had the most access to original artisans' repertoires of the past. His last name means a skillful designer. The ornamental qualities of the geometric constructions in his book and their executions performed on different mediums on the walls, floors, interiors and exteriors of

domes, doors and windows, and many more structures provide a pleasurable journey to the past for all readers.

The 10-Petal Rose Construction: Following the images in Figure 10 from top left to bottom right, divide the right angle $\angle A$ into five congruent angles by creating four rays that emanate from A (angle $\angle COD$ in Figure 9 paper [6] divides a right angle into five congruent angles). Choose an arbitrary point P on the third ray, counter-clockwise, and drop perpendiculars from P to the sides of angle $\angle A$. This results in the rectangle $ABPC$ along with four segments inside this rectangle, each with one endpoint at A and whose other endpoints are the intersections of the four rays with the two sides of PB and PC of rectangle $ABPC$ (the first image in Figure 10). Find O , the midpoint of the third segment created from the third ray. Also, find M , the midpoint of the second segment. Construct two arcs with center A and radii AO and AM . Point N is the intersection of the fourth ray and the smaller arc. Drop a perpendicular from N to AB to intersect the second and third rays to create two more points. Make a parallel line to the first ray from M and also parallel lines to the third ray from M and N to find the intersection points as are illustrated in the second and third images. Using these points and properties of the reflective and rotational symmetries, one is able to make a set of new points that constitute the petals of the rose, as is illustrated in the third and fourth images. Find the intersection of the first ray and the larger arc and drop a new perpendicular line to AB as is illustrated in the fourth image. Extend the segments that constitute the petals and find the intersections of them with segments that are created by the large arc and the rays as is shown in the fifth image. The sixth image shows the final segments for the construction of the 10-petal rose about vertex A . The seventh image in Figure 10 is created by a rotation of 180° about O of the rose part, which is a usual action in most girih to complete the design. The last image in Figure 10 is the complete 10-petal rose girih.

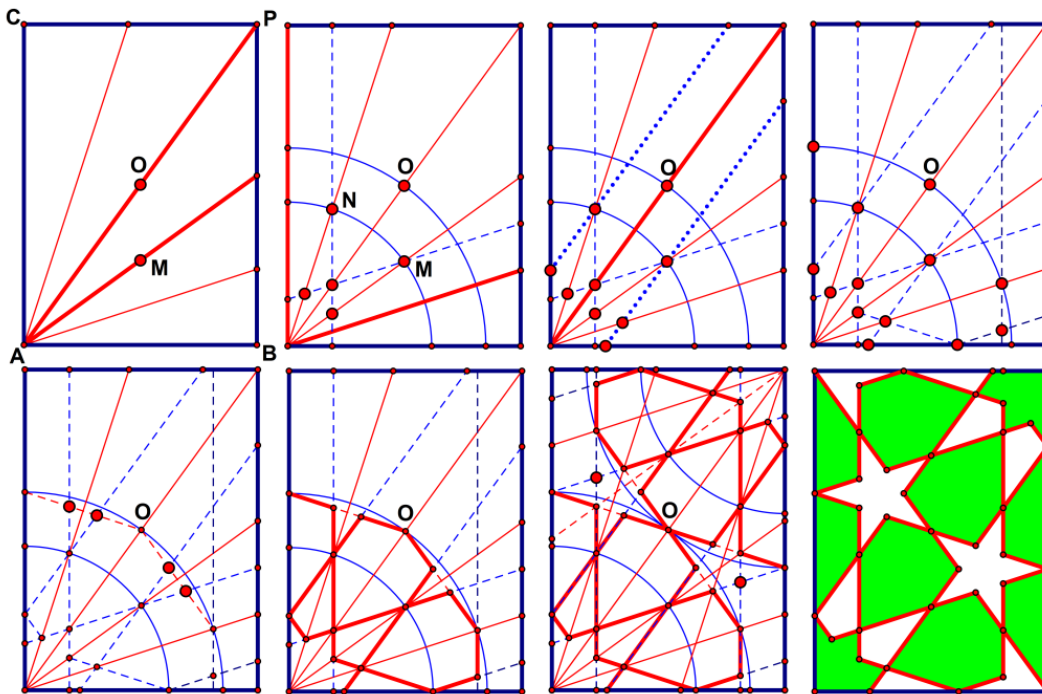


Figure 10: *The composition of 10-petal rose pattern using the Radial Grid Method*

The 12-Petal Rose Construction: To create this girih using the radial grid method, the right angle should be divided into six congruent angles to create a 12-petal rose (the angle bisector of one of the angles of an equilateral triangle creates two 30° angles and repeating the process for one of them creates two 15° angles). In general, the construction should begin by selecting an arbitrary point P on the fourth ray. But

since the two generated girih should be combined as is shown in Figure 10, the two rectangles should have the same size widths. So the process should start by selecting segment AB in the last construction as the width and then finding P by dropping a perpendicular from B to AB to intersect the fourth ray. The rest of the girih construction can be followed step-by-step through images in Figure 11.

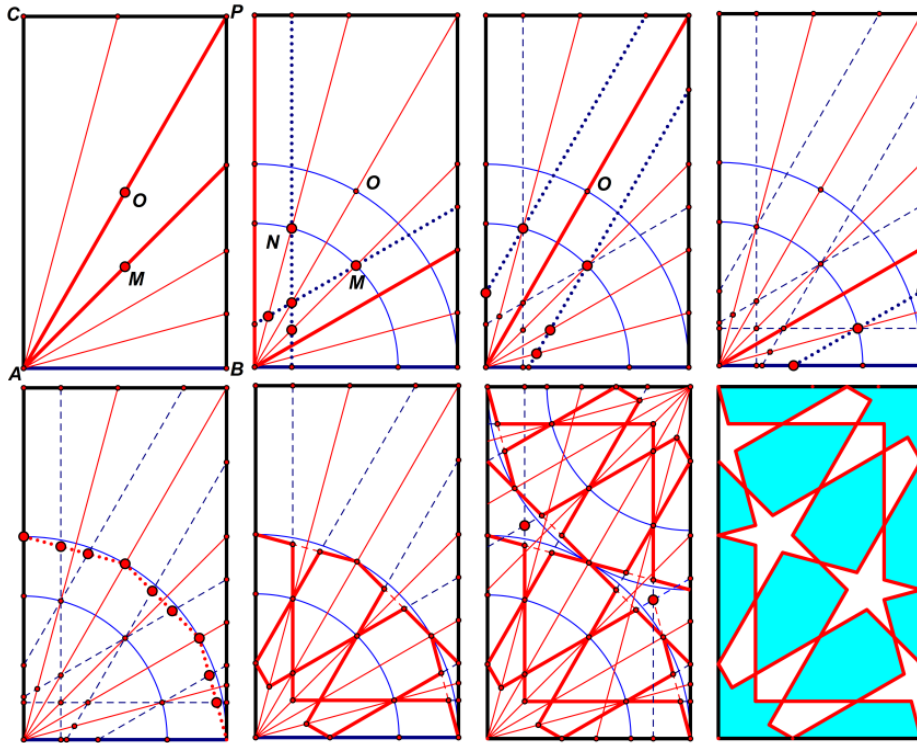


Figure 11: *The composition of 12-petal rose pattern using the Radial Grid Method*

Now following the images in Figure 9 one can pick one half of each girih in a way that the two centers coincide. Figure 12 presents three tessellations. The left one is the 10-petal rose tiling and the right one is the 12-petal rose tiling. The middle image is the 10-12 petal combination tessellation.

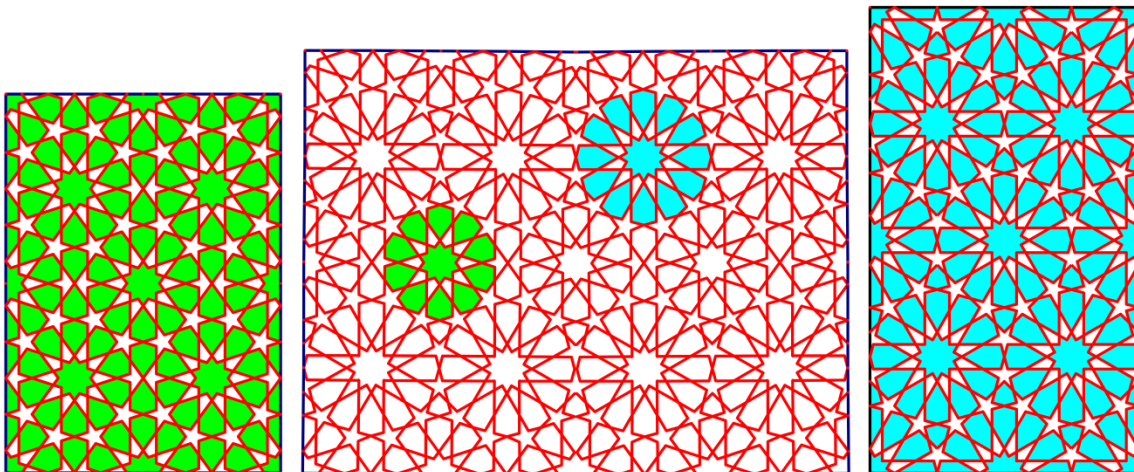


Figure 12: *The 10-petal tiling, the 10-12 petal combination tiling, and the 12-petal tiling*

Comparing the two left and right images in Figure 9, the exact image from the treatise and the final construction by the author, one notices extra lines and arcs in the original image in the treatise. It seems that these are the steps that the author of the treatise has taken to construct the girih. However, it is very possible that the author was only a person who recorded interesting girih and sometimes tried to justify the constructions by making some extra lines and arcs. After examining these extra lines and applying the suggested method written in the document, it is not difficult to come to the conclusion that this method does not work and is wrong. The aforementioned radial grid method that M. Maher-al-Naghsh presented is very organic, simple, and effective, in that it can be claimed that this was the method that was used to create the 10-12 petal rose girih originally. It is very possible that the author of the treatise was a mathematician who was trying to justify the steps of the girih construction and not an artisan, who was in general very familiar with construction methods available at that time.

3.3. The Last Girih. The number of pages of the *Interlock* treatise is twenty. In the last page of the document one can find the last girih as it appears in the left image in Figure 13. The right image in Figure 13 shows a tiling that has been composed based on this girih. The images in Figure 14 present the step-by-step instruction for the construction of the girih.

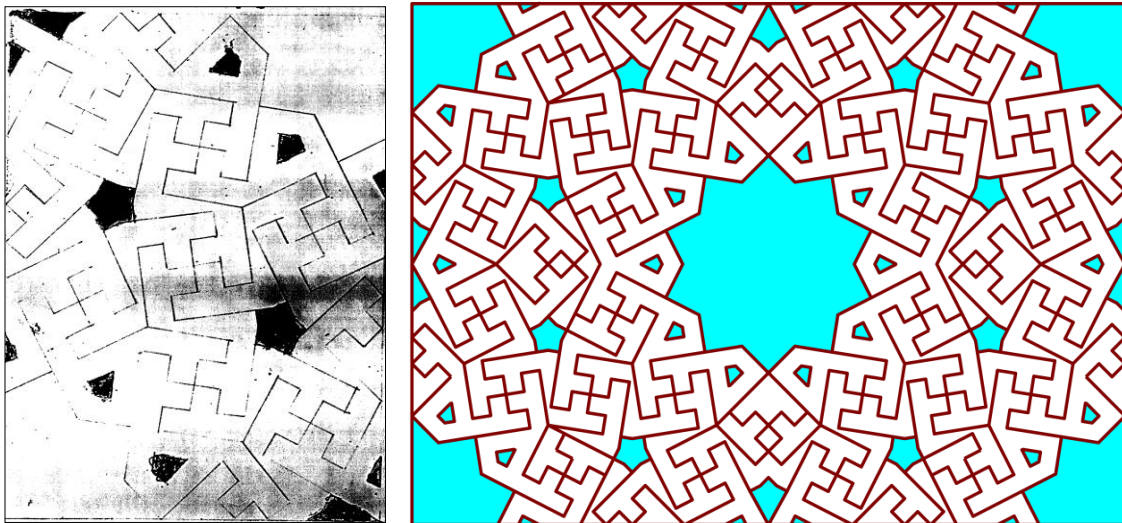


Figure 13: (Left) *The last girih in the treatise, and (Right) The tessellation made from this girih*

Divide the given right angle into five congruent angles. Select an arbitrary point P on the second ray. Make a perpendicular line to this ray at P to intersect the first and third rays at A and B respectively. Make a circle with center P and radius PA . This circle meets the second ray at C . Angle $\angle ACB$ is a right angle. Continue this process to make a quarter of a decagram as in the second image. The third image exhibits how one can make petals for this rose by extending the obtained segments in the previous image. The fourth image shows how one can make squares using the sides of the petals. The next image presents another congruent square whose diagonal is a part of the third ray. Find the center of this square and call it O . Now by rotating the constructed part in the fifth image by 180° , about O one can generate a girih as in the sixth image. The seventh image shows how a person can divide the square in the middle into sixteen small squares and then keep some sides of the small squares and erase the rest to make the design in the middle of the image. The next two images demonstrate how one can continue this process of making designs on all other squares to complete the process.

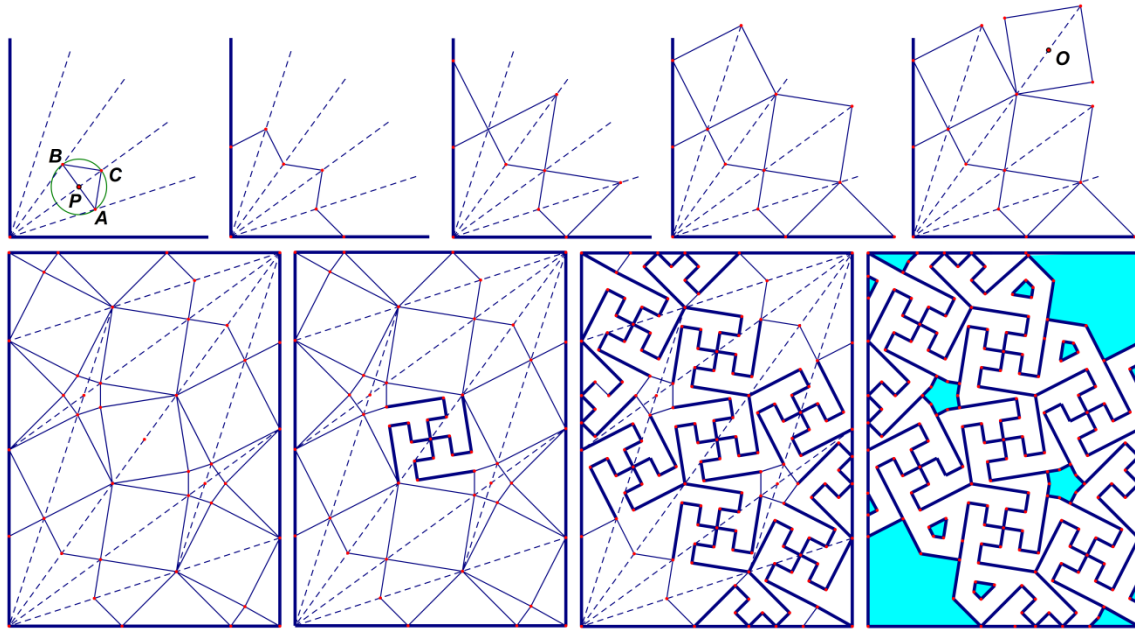


Figure 14: The girih making process for the girih in Figure 19.

4. Conclusion

The treatise *Interlocks of Similar or Complementary Figures*, even though many different parts of it are not clear and comprehensive, and despite the fact that this is the only document from the past that shows how girih is composed, establishes the fact that there was a strong relationship between mathematicians and artisans in medieval Persia. This relationship, as can be observed from the treatise, was purely based on the knowledge of mathematics and not on any system of ideology or belief. Even though the treatise is old, many ideas for puzzle and girih making are fresh and can be used as parts of creativity and learning projects and workshops. The treatise provides readers with opportunities to engage in some practical activities, to develop new tiling patterns, or create topics for the classroom.

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