

A Skeleton Key for the Platonic Solids

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Abstract

Dance performances in which dancers wield sections of PVC pipe to form a number of polyhedral designs led to the question: what is the largest graph which edge decomposes the “skeletons” of all five Platonic solids? We show that a certain 6-edge tree is the largest such graph, and investigate related questions.

PVC Pipe Polyhedra. The tetrahedron in Figure 1 is composed of a folded octagon, while the cube and octahedron are each composed of six folded paths of length 2. These constructions have been used in dances created by the author and his collaborators. The photo from the author’s dance “Pipe Dreams” shows an octahedron in which each dancer wields three lengths of PVC pipe held together by cord at the two internal vertices labeled a in the diagram on the right. The white pipe tends to stand out in bright stage lighting, though in some dances the pipes are painted with fluorescent paint and black lights make the pipes “pop” even more. The shapes created by the dancers, which might include whimsical designs reminiscent of animals or other objects as well as mathematical forms, seem to appear and dissolve in fluid patterns, usually in time to a musical score.

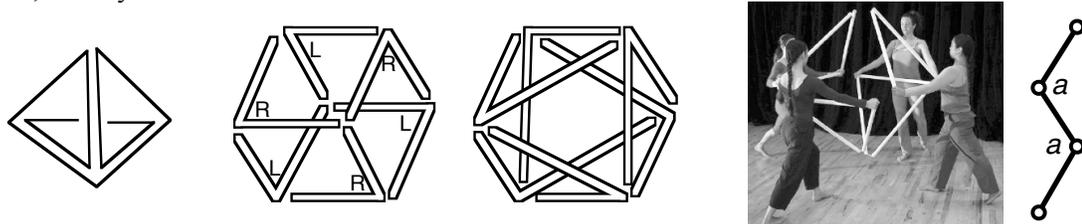


Figure 1: *PVC pipe constructions used in dances.*

Our desire to incorporate polyhedra into dance works led to these constructions, and to similar designs with loops of rope, fingers and hands, and the bodies of dancers. Just as mathematical concepts often suggest artistic explorations for those involved in the interplay between these fields, performance problems may suggest mathematical questions, in this case involving finding efficient and symmetrical ways to construct the skeletons, or edge sets, of the Platonic solids. In [6] the authors showed classroom activities involving making polyhedra with PVC pipe, fingers, and loops of string; George Csicsery documented the latter two of these in a series of short films [1]. In previous papers the author investigated modular constructions of the Platonic solids in a manner reminiscent of modular origami, and found that six seemed to be a kind of “magic” number for these constructions: in [7] the author showed how to construct the five Platonic solids with six loops of three colors, and in [5] with length six PVC pipe modules and also with the bodies of six dancers.

Decompositions of graphs by trees have been popular recently. A decomposition is a partition of the edges into multiple copies of a particular graph which may overlap only at the vertices, such those of the cube and octahedron in Figure 1. The decomposition of the hypercube Q_n into copies of paths and trees

has gained interest due to the construction of massively parallel-processing computers whose architecture is that of a hypercube [3]. The complete graph K_n has n vertices, each adjacent to every other vertex. In 1964, Gerhard Ringel conjectured that the $m(2m+1)$ edges of the complete graph K_{2m+1} may be partitioned into $2m+1$ copies of any tree with m edges, and this problem is equivalent to what is known as the unsolved graceful tree conjecture [4], which says that every tree may receive a “graceful” labeling. However, the author’s interest in decomposing the edges of polyhedra originated in the use of polyhedra in dance choreography.

The largest graph multiple copies of which can be folded so as to decompose the skeletons of each of the five Platonic solids was shown in [5], but the complete proof that this graph is unique was omitted. The following outline of the proof will be repeated from [5] and the complete proof will be included in this paper and in its appendix. This decomposition must avoid overlapping edges such as occur in the tetrahedron in Figure 1. Such a graph would necessarily have a number of edges that is a divisor of the 6, 12, and 30 edges of these five polyhedra, and also cannot have vertices of degree greater than 3, since the tetrahedron, cube, and dodecahedron only have degree 3 vertices. The cube and dodecahedron rule out the use of a 3-cycle or 6-cycle, since those polyhedra have vertices of odd degree, and each vertex of such a cycle would add two degrees to any of their vertices. Also, any such decomposing graph cannot contain a cycle of length less than 5, since that is the smallest cycle in the dodecahedron. This limits the possibilities to the nine 2-, 3- and 6-edge tree graphs shown in Figure 2.

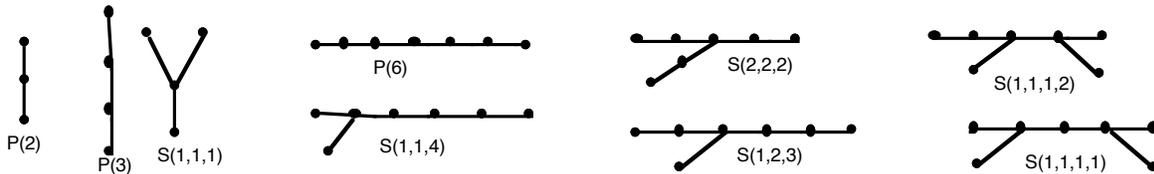


Figure 2: Possible graphs decomposing the Platonic solids.

For convenience, these graphs have been labeled either as a path $P(n)$ with n edges, or as $S(a,b,c,d)$, where $a,b,c,$ and d indicate the lengths of the appendage paths from degree three vertices. Two copies of the “3-star” $S(1,1,1)$ do not fold into the tetrahedron, though this graph does decompose the cube, octahedron, and icosahedron. The decompositions of the cube and octahedron can be nicely modeled using the thumb, first, and second fingers of four hands [6]. The dodecahedron cannot be decomposed into 3-stars, since every pentagon in the dodecahedron would require edges from three 3-stars, two of whose edges would then overlap. There are many decompositions of all five Platonic solids using $P(2)$, and Figure 3 (right) shows decompositions of the icosahedron and dodecahedron using $P(3)$. If the two instances of $P(3)$ shown as A and B are joined at one vertex, we get decompositions using $S(1,2,3)$, which are also shown decomposing the tetrahedron, cube and octahedron in the left of Figure 3 (the vertices labeled x in the icosahedron are identified as one vertex.). $P(6)$ will not fold into the tetrahedron since $P(6)$ has only two odd degree vertices, while the tetrahedron has four, and the other vertices of $P(6)$ are even, of degree 2. Similarly, five copies of $P(6)$ provide only 10 odd degree vertices, not enough for the 12 and 20 odd degree vertices in the dodecahedron and icosahedron.

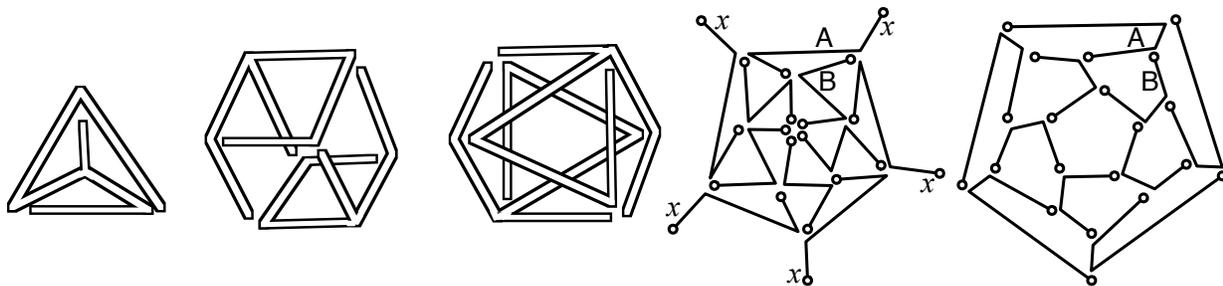


Figure 3: Decompositions of the Platonic solids with $S(1,2,3)$ and $P(3)$

$S(1,1,1,1)$ will not decompose the tetrahedron for the following reason: every pair of vertices in the tetrahedron is adjacent to five edges, and every pair of vertices is adjacent to each other; however, the two degree 3 vertices in $S(1,1,1,1)$ are adjacent to a total of 6 edges. $S(1,2,3)$ does decompose all five, as shown in Figure 3.

Theorem. $S(1,2,3)$ is the unique six-edge graph which edge-decomposes all five Platonic solids.

We have already seen that $P(6)$ and $S(1,1,1,1)$ do not work, while $S(1,2,3)$ does. Interestingly, $S(2,2,2)$, $S(1,1,4)$, and $S(1,1,1,2)$ decompose all but the dodecahedron. The proof of this is somewhat lengthy and is in this paper’s appendix, but we will first point out some interesting facts about these decompositions.

The tables in Figure 4 show which of the six-edge trees in Figure 1 above decompose which of the Platonic solids. Certain “modules” composed of six edges make the decompositions more uniform. For example, the graph  composed of a triangle with a pendant edge (having a degree one vertex) and another pendant path of length two may be formed by $S(1,2,3)$, $S(1,1,4)$, $S(1,1,1,2)$, or $S(2,2,2)$, and this module easily decomposes both the octahedron and the icosahedron, showing that these four trees also decompose these two Platonic solids. For example, in the icosahedron in Figure 1 the paths A and B of length 3 combine to give this module, five copies of which decompose the icosahedron; two copies of this graph also decompose the octahedron in Figure 1. Two copies of the graph  are used in Figure 1 to construct the skeleton of the cube, showing that $S(1,2,3)$, $S(1,1,4)$, and $S(1,1,1,2)$ decompose it. This construction is used as an audience interaction puzzle in a number of our shows: an audience member is brought on stage and challenged to bend a copy of this graph made with PVC pipe and wire into the shape of the tetrahedron. (If it is shown to the volunteer with all angles as right angles, the task is a little more puzzling!) Those entries in the table labeled “YES” indicate another method is needed to decompose the polyhedron with the tree in question, and “NO” indicates the decomposition is impossible.

Type	$S(1,2,3)$	$S(1,1,4)$	$S(1,1,1,2)$	$S(2,2,2)$	$S(1,1,1,1)$
		NO		NO	
					NO
				NO	NO

	$P(6)$	$S(1,1,1,1)$	$S(2,2,2)$	$S(1,1,4)$	$S(1,1,1,2)$	$S(1,2,3)$
Tetrahedron	No	No	Yes			
Cube	No	Yes	Yes			
Octahedron	Yes	Yes				
Icosahedron	No					
Dodecahedron	No	No	No	No	No	Yes

Figure 4: *Decompositions of Platonic Solids with modules from 6-edge trees*

A variety of other polyhedra are decomposable by these trees. For example, the edges of the rhombic dodecahedron may be partitioned into four copies of the 4-cyle with two pendant edges, as in Figure 5, showing it is decomposable by $S(1,2,3)$, $S(1,1,4)$, and $S(1,1,1,2)$. (Here the vertices labeled A are

identified.) Also in Figure 5 we see the rhombic triacontahedron similarly decomposed into copies of the same module, so it is also decomposable by the same trees. Its five pendant vertices are also identified. The 13 Archimedean solids have the following numbers of edges: 18, 24, 36, 48, 60, 72, 90, 120, 150, and 180. Since their greatest common divisor is also six, any graph that decomposes all of them must have no more than six edges. In fact, all 13 are decomposable by $S(1,2,3)$ – these are shown in [8]. Curiously, and perhaps a clue to its versatility, $S(1,2,3)$ is identical in shape to the Dynkin Diagram E_7 , which is often said to be ubiquitous in the study of Lie Groups and associated algebraic systems.

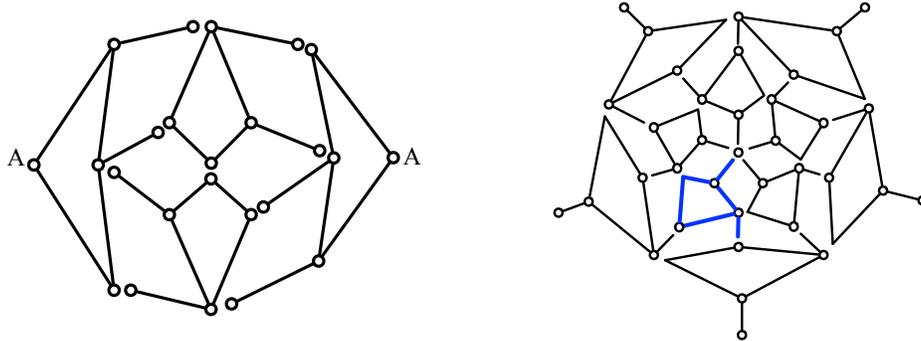


Figure 5: *Decompositions of the rhombic dodecahedron and rhombic triacontahedron*

$S(1,2,3)$ also decomposes the edges of all the regular and semi-regular or Archimedean tilings of the plane, two of which are shown in Figure 6. If used in a dance, each $S(1,2,3)$'s would most likely need to be manipulated by two dancers, and we have not yet experimented with this construction, as the dodecahedron and icosahedron would take ten dancers in complex formations. However, we have been creating dances during the past year using a downward directed camera hung above the performance space, with the image projected on a screen for audience viewing, and this opens possibilities for a number of dancers using their torso and arms to create tiling patterns, reminiscent of the work of Busby Berkeley. Also in Figure 6 is a human figure seen from above extending one arm forward and one arm backward to create an approximation of $P(3)$, two of which make up $S(1,2,3)$. Diagrams of all these tessellations are shown in [8]. These figures might also be the basis for entertaining animations. Another PVC pipe construction which we have actually used in a dance is a hexagon with six pendant edges, discussed in [5]. It was used in *Harmonious Equations*, directed by Keith Devlin, in 2009 [2], in the section on Euler's formula relating the numbers of vertices, edges, and faces of polyhedra.

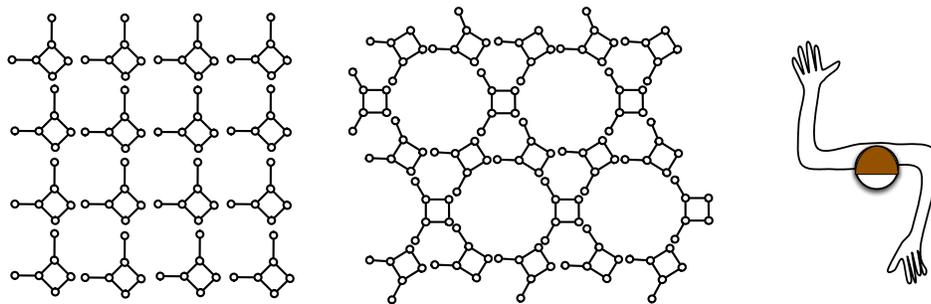


Figure 6: *Truncated square and truncated trihexagonal semiregular tessellations, and a human "P(3)"*

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Appendix: Proof of the theorem.

We now show that $S(1,1,1,1)$, $S(2,2,2)$, $S(1,1,4)$, and $S(1,1,1,2)$ will not decompose the dodecahedron.

$S(1,1,1,1)$. Though $S(1,1,1,1)$ does not decompose the tetrahedron, we show here it also does not decompose the dodecahedron. Every embedding of $S(1,1,1,1)$ in the skeleton of the dodecahedron has at least four of its edges in the same pentagonal face, since the two central edges of $S(1,1,1,1)$ must lie in a common pentagon, and two of the other $S(1,1,1,1)$ edges must be edges of that pentagon as well, as in Figure 7(a). Since every edge of $S(1,1,1,1)$ has a vertex that is of degree 3, the fifth edge e of the pentagon shown in Figure 8 cannot be an edge of another $S(1,1,1,1)$, since the end vertices of e already have degree 1 contributed by the first copy of $S(1,1,1,1)$. Therefore the dodecahedron graph cannot be decomposed into copies of $S(1,1,1,1)$.

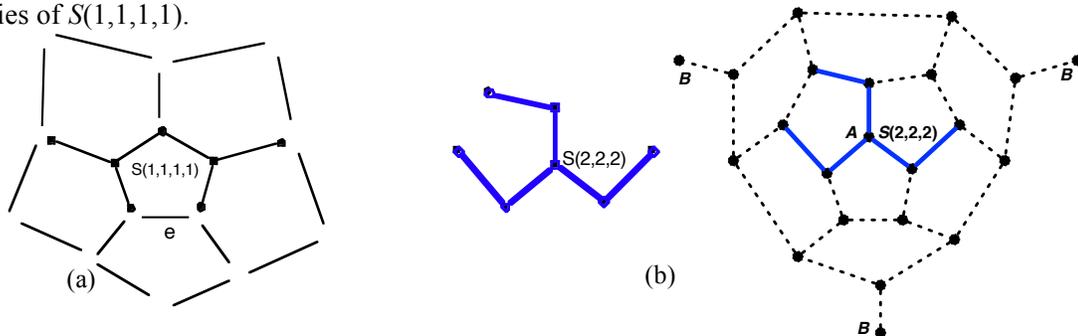


Figure 7: Impossibility of decomposing the dodecahedron with $S(1,1,1,1)$ or $S(2,2,2)$

$S(2,2,2)$. Without loss of generality, the center vertex A of $S(2,2,2)$ is shown at the center of the Schlegel diagram of the regular dodecahedron in Figure 7(b); the opposite vertex B is shown three times in order to simplify the drawing, but all three are identified as a single vertex. Then no other copy of $S(2,2,2)$ may be placed so that its central degree 3 vertex is at one of the three vertices at distance 1 from A , since two edges at those vertices are already in use by the first copy of $S(2,2,2)$. Also no copy of $S(2,2,2)$ may be placed with its central vertex at one of the six vertices at distance 2 from A , since there will then be overlap along one of the edges belonging to the first copy of $S(2,2,2)$ placed at A . If another copy of $S(2,2,2)$ has its central vertex at B , then all other vertices will be distance 2 or less from either A or B , and so no further copies of $S(2,2,2)$ will fit on the dodecahedral graph. Thus the four copies of $S(2,2,2)$ other than that centered at A must all have their central degree 3 vertices on the 9-gon whose vertices are at distance 3 or 4 from A . Since each such $S(2,2,2)$ precludes any other copies with their central vertices at distance 1 or 2, there is only room on this 9-gon for two such copies of $S(2,2,2)$. Therefore at most three

copies of $S(2,2,2)$ will fit on the dodecahedron without overlap, and its 30 edges cannot be decomposed into five copies of $S(2,2,2)$.

$S(1,1,4)$. Figure 8 shows the four distinct ways in which $S(1,1,4)$, indicated by darker edges, might be embedded in the skeleton of the dodecahedron. We first show that such a decomposition by $S(1,1,4)$ cannot contain a type I embedding and must have at least one type III embedding, then show that a single type III embedding leads to a contradiction.

Type I. $S(1,1,4)$ has three vertices of degree 1 and one of degree 3, while the dodecahedron has 20 vertices of degree 3. In a successful decomposition of the dodecahedron by five copies of $S(1,1,4)$, having a total of 20 odd degree vertices, the $S(1,1,4)$ s must contribute one odd degree vertex per dodecahedral vertex, so it is impossible for such a decomposition to have a vertex at which three degree one vertices of the composing $S(1,1,4)$ s meet, such as the vertex labeled v in the Type I diagram. Therefore there can be no type I copies of $S(1,1,4)$ in such a decomposition.

Type II and type IV. A parity argument can be used to show that there is no decomposition of the dodecahedron by five $S(1,1,4)$ s consisting only of types II and IV. Figure 8 shows the number of edges that each type I, II, and III $S(1,1,4)$ contributes to each pentagonal face. Each type II and IV embedding of $S(1,1,4)$ contributes an odd number of edges to two pentagonal faces. A decomposition consisting only of type II and IV would thus contribute an odd number of edges to at most ten of the twelve faces. However each of the pentagonal faces has an odd number of edges, so none can be composed entirely of even numbers of edges contributed by the $S(1,1,4)$ s, and thus each of the twelve must be one of the ten faces receiving odd numbers of edges, which is impossible. Therefore any successful decomposition by $S(1,1,4)$ must contain at least one type III embedding.

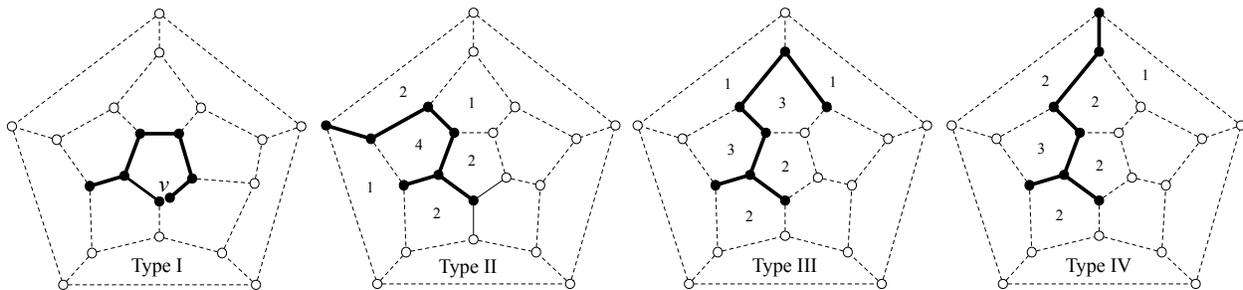


Figure 8: Four types of embedding of $S(1,1,4)$ in the dodecahedron.

Type III. The remaining analysis shows that any successful decomposition by $S(1,1,4)$ cannot contain a single type III copy. In the discussion of type I we saw that any decomposition by $S(1,1,4)$ must combine each degree one vertex of an $S(1,1,4)$ with a degree 2 vertex of another $S(1,1,4)$. Thus the two edges labeled c in the diagrams for the type III embeddings in Figure 9 must be part of the same copy of $S(1,1,4)$, and similarly for the two edges labeled d and also the two edges labeled e .

The three diagrams in Figure 9 show an embedding of a type III $S(1,1,4)$ in bold, with the two c edges, which must be part of the same copy of $S(1,1,4)$, continued in three possible ways. In (A) the edges u and v are in the same copy of $S(1,1,4)$ as the two c edges. In (B) only edge v is in the same copy of $S(1,1,4)$ as the c edges, and in (C) neither u nor v is in that copy. If in (A) the $S(1,1,4)$ containing the c edges also contains edges x and y then this $S(1,1,4)$ will be type I, which we have seen cannot be completed to a full decomposition. If it contains x and z then the $S(1,1,4)$ containing edge y cannot also contain edge s , leaving edges y or s isolated and not able to be part of another $S(1,1,4)$. If it contains edges w and t then the only possible continuation of the d edges will contain only one of the e edges, which is not possible.

In Figure 9(B), if the $S(1,1,4)$ containing the c edges also contains edges x and y , then it will also include edge z , and the $S(1,1,4)$ containing edge u will isolate edge s . If instead it contains edge w , then it

will include only one of the d edges. In (C) the two c edges are at the end of the length 4 path section of its $S(1,1,4)$. If this $S(1,1,4)$ also includes edges x and y then it will also include u and be type I. If it contains instead edges x and z then the $S(1,1,4)$ containing v will isolate the edge y . If it contains instead the edges w and t , then as before it will be impossible to continue the d edges so as to complete the decomposition. Therefore there is no decomposition of the dodecahedral skeleton that includes any type III embeddings of $S(1,1,4)$, and so finally an $S(1,1,4)$ decomposition is not possible.

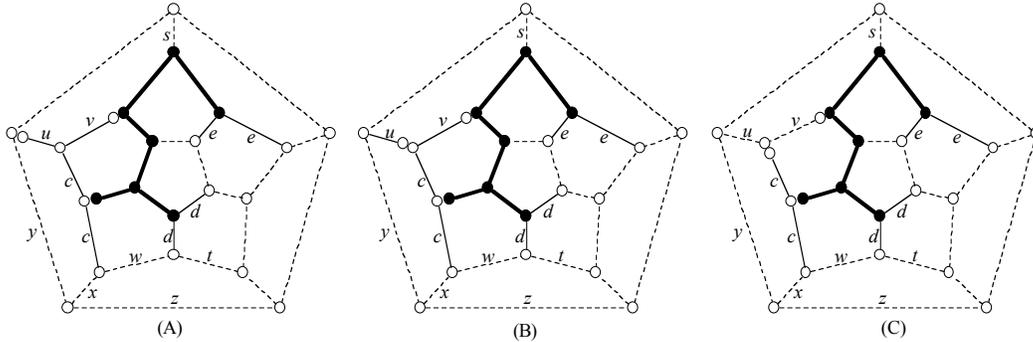


Figure 9: Type III embeddings of $S(1,1,4)$.

$S(1,1,1,2)$. Figure 10 shows the two distinct ways that $S(1,1,1,2)$ can be oriented within the dodecahedron, disregarding symmetries, labeled type A and type B. Type A contributes 1, 2, 2, 3, and 4 edges to the adjacent pentagonal faces, and type B contributes 1, 2, 3, 3, and 3 edges to the adjacent faces. Note that in all cases the faces which have 2 edges contributed by $S(1,1,1,2)$ share one end vertex with the edge e , which we call the *spine* of the $S(1,1,1,2)$, which joins $S(1,1,1,2)$'s two degree 3 vertices, a fact that we will use later.

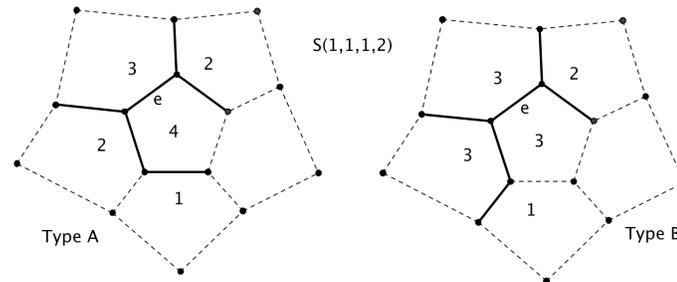


Figure 10: Two types of $S(1,1,1,2)$ embedding in the dodecahedron.

Suppose a decomposition of the dodecahedron using $S(1,1,1,2)$ uses a copies of type A and b copies of type B, where necessarily $a + b = 5$. The total number of 1-edge contributions to the 12 pentagonal faces, as described above, will be $a + b = 5$, and similarly the number of 2-, 3-, and 4-edge contributions will be $2a + b = 5 + a$, $a + 3b = 15 - 2a$, and a , respectively. There are six ways in which the 1-, 2-, 3-, and 4-edge contributions can combine within one of the faces, disregarding symmetries, shown in Figure 11, where vertices from distinct copies of $S(1,1,1,2)$ which merge at one vertex of a pentagon are shown as empty circles.

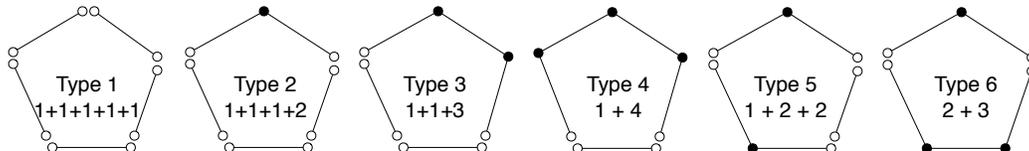


Figure 11: Ways that length 1, 2, 3, and 4 segments can combine within one face.

Let the number of type j faces be x_j ; then the following equations summarize the edge-counting constraints of the x_i , where the i -th equation gives the total number of length i segments in all the faces.

$$\begin{array}{l}
 5x_1 + 3x_2 + 2x_3 + x_4 + x_5 = 5 \\
 x_2 + 2x_5 + x_6 = 5 + a \\
 x_3 + x_6 = 15 - 2a \\
 x_4 = a
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{l}
 x_1 = x_5 + x_6 - 8 \\
 x_2 = a - 2x_5 - x_6 + 5 \\
 x_3 = -2a - x_6 + 15 \\
 x_4 = a
 \end{array}$$

This set of equations has the solutions shown on the right, where a , x_5 , and x_6 are free variables. Note that the sum of the four row equations of this solution gives $\sum_1^6 x_i = 12$, or that there are a total of twelve faces, which is therefore a redundant equation; additional constraints are that a , b , and each x_i are non-negative. Using the non-negativity of the x_i , the first solution equation gives $8 \leq 8 + x_1 = x_5 + x_6$ and the second gives $2x_5 + x_6 \leq 2x_5 + x_6 + x_2 = a + 5$, and combining them we have

$$(1) \quad 8 \leq x_5 + x_6 \leq 2x_5 + x_6 \leq a + 5,$$

or $3 \leq a$. We will examine the three cases $a = 3, 4$, and 5 .

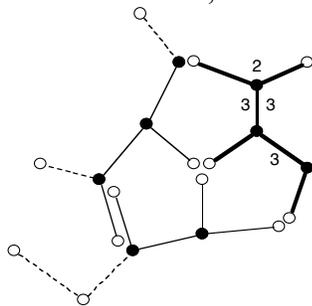
(I) $a = 3$. Equation (1) gives $8 \leq x_5 + x_6 \leq 2x_5 + x_6 \leq a + 5 = 8$, which implies that $x_5 = 0$ and $x_6 = 8$. The other solutions above for the x_i then give $x_1 = x_2 = 0$, $x_3 = 1$, and $x_4 = a = 3$.

(II) $a = 4$. The equation for x_3 implies that $0 \leq x_3 = 15 - 2a - x_6$, or $x_6 \leq 15 - 2a = 7$, and equation (1) gives $8 \leq x_5 + x_6 \leq 2x_5 + x_6 \leq 9$; the only possible solution has $x_6 = 7$ and $x_5 = 1$. These then determine the values $x_1 = x_2 = x_3 = 0$ and $x_4 = a = 4$.

(III) $a = 5$. Equation (1) implies that $8 \leq x_5 + x_6 \leq a + 5 = 10$, and the equation for x_3 implies that $0 \leq x_3 = 15 - 2a - x_6$, or $x_6 \leq 15 - 2a = 5$. In order that $8 \leq x_5 + x_6$ we must have $x_5 \geq 3$, but then we cannot also have $2x_5 + x_6 \leq 10$, so there are no solutions with $a = 5$.

Now we check the two possible solution types I and II geometrically. First consider case II. Because $x_3 = 0$, any pentagonal face with a 3-edge segment contributed by a copy of $S(1,1,1,2)$ must have those 3 edges matched by a 2-edge segment from another copy of $S(1,1,1,2)$. The case B embedding of $S(1,1,1,2)$ has two adjacent pentagons, separated by a pendant edge, to which it contributes 3 edge segments, as shown in Figure 12. Because the 2-edge faces of the two pairs of $S(1,1,1,2)$ edges matching these 3-edge segments must be adjacent to the central spines of their $S(1,1,1,2)$ s, there will be a pentagonal face bounded by them in which two of their edges overlap, since two degree 3 vertices of different $S(1,1,1,2)$ s are adjacent in this pentagon. Therefore solution II is not possible.

In solution I, there are $5 - a = 2$ type B embeddings of $S(1,1,1,2)$ s, but $x_3 = 1$. So one of the type B $S(1,1,1,2)$ s again has two adjacent pentagonal faces to which it contributes 3-edge segments that must be matched by 2-edge segments of two additional $S(1,1,1,2)$ s, which again produces a contradiction. So solution I is also not possible, and there are no decompositions of the dodecahedron by five $S(1,1,1,2)$ s.



Therefore $S(1,2,3)$ is the largest graph that decomposes all five Platonic solids. Further work on this problem might articulate which trees decompose which Archimedean solids or other groups of polyhedra, and explore the symmetries of these decompositions and how they relate to planar tilings.

Figure 12: Case B for $S(1,1,1,2)$