Fun with Whirls

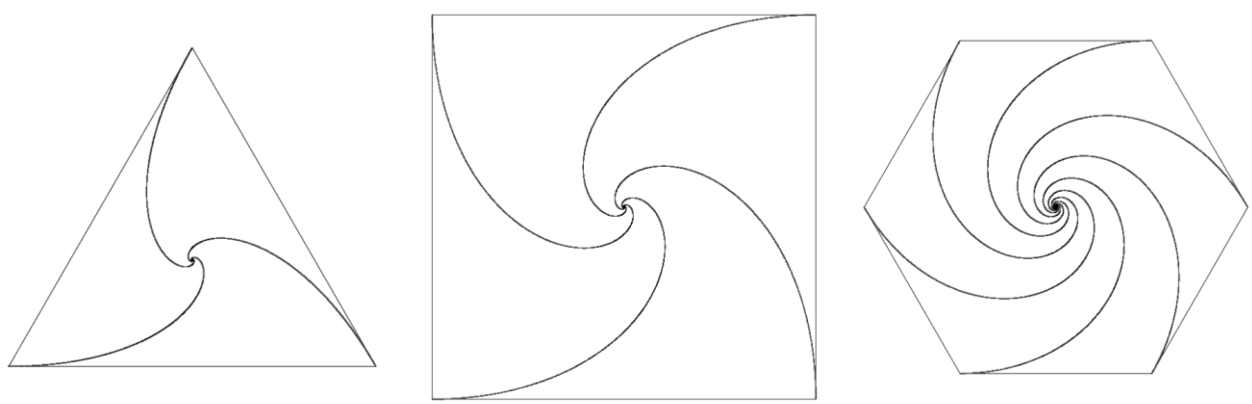
Kerry Mitchell
3833 North 30th Street
Unit 247
Phoenix, AZ 85016, USA
E-mail: lkmitch@gmail.com

Abstract

This work examines whirls, figures of nested polygons that approximate pursuit curves, and how they can be used in visual art. A general approach to creating whirls from any base polygon is presented. It is then used to explore an assortment of whirls, including variations in: base polygon, numbers of vertices, and the relative distance between old and new polygons. Methods for rendering whirls are discussed and several finished pieces are shown.

What is a Whirl?

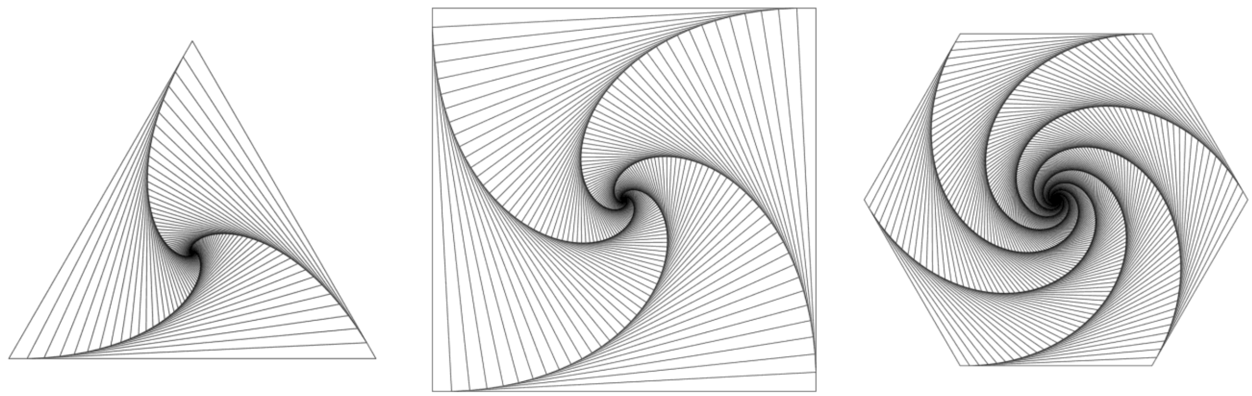
A whirl is a discrete approximation to a pursuit curve. A pursuit curve, in turn, is a curve generated by the motion of a point A in pursuit of another moving point, B. That is, the velocity of point A is always directed toward point B. The mice problem [1] gives examples of pursuit curves. In the mice problem, n mice begin in the corners of an n-sided regular polygon. Each mouse continuously moves toward its nearest neighbor, in the counter-clockwise direction, at the same speed. The paths traced out by the mice are logarithmic spirals, converging at the center of the polygon. Figure 1 shows such curves for the cases of an equilateral triangle, a square, and a regular hexagon.



a) *Equilateral triangle* b) *Square* c) *Regular hexagon*

Figure 1: *Logarithmic spirals in the mice problem.*

For a regular polygon and mice (or points) moving at constant speeds, the continuous analytical solution can be found (see, for example, Wolfram [2]). However, given irregular polygons or non-constant speeds, the problem is much more difficult. Whirls present approximations to pursuit curves, by showing piecewise linear paths instead of smooth curves. In essence, the pursuit curve equations are integrated for discrete time steps. Conveniently, this process has a geometric solution—iteratively constructing a series of smaller polygons inside of the base polygon. Each subordinate shape is smaller than, and rotated relative to its parent, with the corners of the child shape lying along the sides of the parent. The result is a series of nested polygons whose corners approximate the pursuit curve. Figure 2 shows whirls for the same cases as in Figure 1.

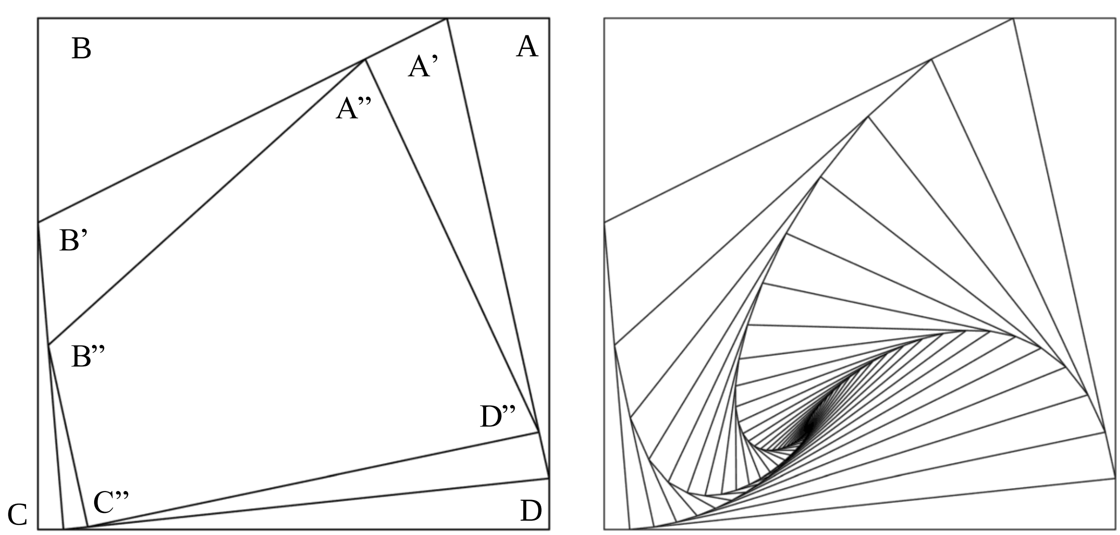


a) *Equilateral triangle* b) *Square* c) *Regular hexagon*

Figure 2: *Whirls in regular polygons.*

General Whirls

The first panel in Figure 3 illustrates the construction process for a general whirl. In this example, the base polygon is a square. Its corners are the points A, B, C, and D. At the first iteration, a new quadrilateral is formed, with corners A', B', C', and D'. Likewise, at the second iteration, the new quadrilateral has corners A'', B'', C'', and D''. The point A' is determined by moving along side AB a relative distance d_A. If $d_A = 0$, then A' lies exactly atop of A. If $d_A = 0.5$, then A' is halfway between A and B, and if $d_A = 1$, then A' lies exactly atop of B. In this example, $d_A = 20\%$, $d_B = 40\%$, $d_C = 5\%$, and $d_D = 10\%$. In the second panel of Figure 3, the process has been continued for 100 iterations, showing the completed whirl.

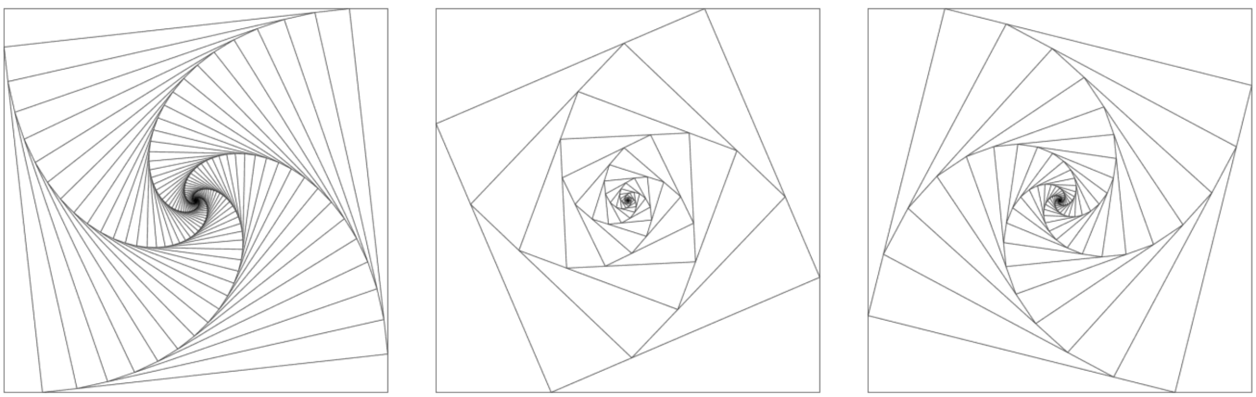


a) *Determining corner points* b) *Final whirl*

Figure 3: *Construction of a general whirl.*

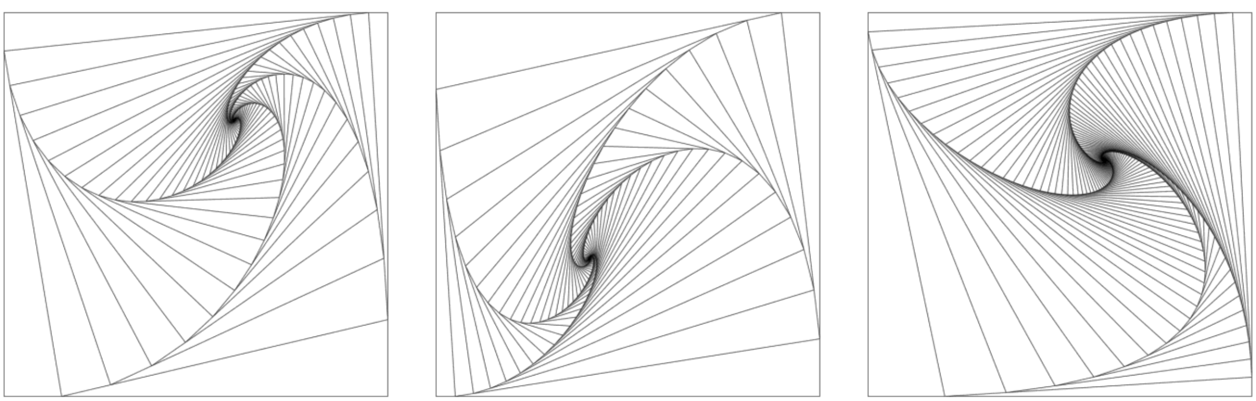
Typically, whirls in the literature have the same value of d for all corners, corresponding to the mice problem with each mouse moving at the same speed. Varying the value of d can have interesting effects; see Figures 4 and 5. In Figure 4, d is the same for all four corners, but varies from 10% to 80%. In this construction, the whirls turn counter-clockwise (from the outside in) for $d < 0.5$. At exactly $d = 0.5$, there is no sense of turning (the corner points align along a straight line), and for $d > 0.5$, the direction changes to clockwise. For d values closer to 0 or 1, more iterations are required for the whirl to converge to the center point and there is a greater sense of apparent shading as the sides of subsequent shapes lie closer together. However, if d is the same for all corners, the curves all converge to the center of the base shape.

Figure 3 shows one example of a case in which d varies across the corners; Figure 5 shows some additional examples. The exact values of d are listed in the caption, from the upper right corner counter-clockwise to the lower right. Generally, as the variation in d increases, the convergence point moves further away from the center of the base shape. Using values closer to 0 or 1 along with values closer to 0.5 gives variation in the apparent shading of the image.



a) *d = 10%* b) *d = 30%* d) *d = 80%*

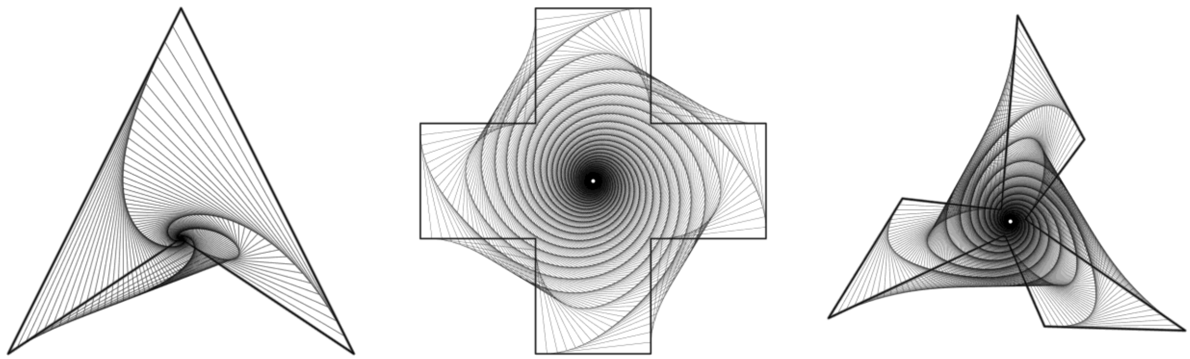
Figure 4: *Effect of various relative distances.*



a) *d = 5%, 10%, 15%, 20%* b) *d = 10%, 20%, 5%, 15%* c) *d = 5%, 5%, 20%, 5%*

Figure 5: *Effect of different relative distances with the whirl.*

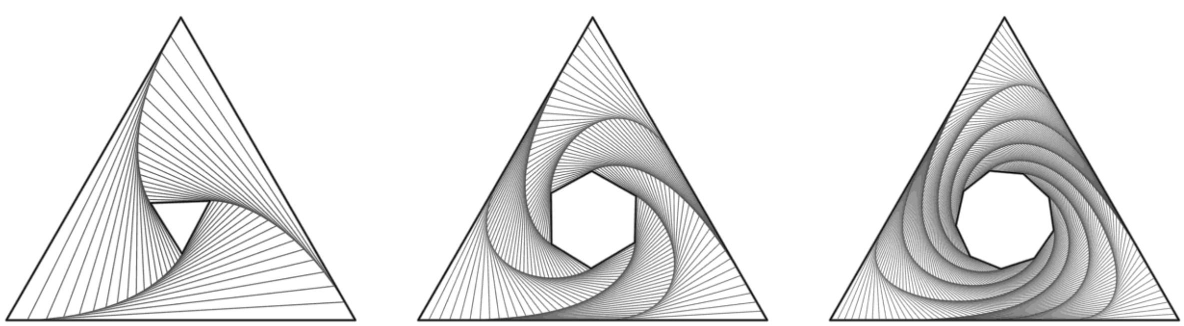
The examples shown thus far have used regular polygons for the bases. A regular polygon is convex and has equal side lengths and corner angles. If irregular polygons are used instead, then interesting things can happen. The various side lengths (distances between corner points) give the image a sense of shading; shorter sides give rise to darker regions and longer sides, to more open areas. Also, if the polygon is concave instead of convex, then the whirls extend outside of the base shape. Figure 6 shows three examples; in each, the bold line shows the base polygon and the value of d is the same for all corners. In the first example, the shape of the dart was chosen such that the convergence point exactly aligned with one of the corners. The cruciform in the second example is a dodecagon (12 sides) and as is often the case with many-sided bases, the numerous curves disguise the original shape. A similar effect occurs with the nonagon (9 sides) in the third case.



a) *Dart* b) *Cruciform* c) *Nonagon*

Figure 6: *Whirls with concave base polygons.*

Another way to employ irregular base polygons is to start with a regular polygon and then add additional points along its sides. This can be helpful when using the whirls in a tiling, where it may be important that they do not extend beyond the sides of the base polygon. Using more points increases the number of curves, as the subsequent polygons have the same number of sides as the number of points used. Figure 7 shows this for the case of an equilateral triangle base. The left panel is the standard whirl, using the three corner points. The iteration is halted midway and an intermediate triangle is highlighted in bold. In the middle panel, three points have been added, at the midpoint of each side. Now, there are six curves and the intermediate polygon is a hexagon. Finally, the base in the right panel has nine points, the three corners and two points equally spaced on each side. This results in nine curves and the subsequent shapes are nonagons.



a) *Three points* b) *Six points* c) *Nine points*

Figure 7: *Using additional points on the sides of the base polygon.*

When using extra points in addition to the base polygon's corner points, the locations of the points can make a big difference in the look of the final image. Figure 8 demonstrates this technique, employing eight points along the sides of a base square. There are now eight curves instead of four, and the subsequent polygons are octagons. In the figure, black dots around the periphery of the squares indicate the locations of the points. In each case, the d value was the same for all points, 10%.

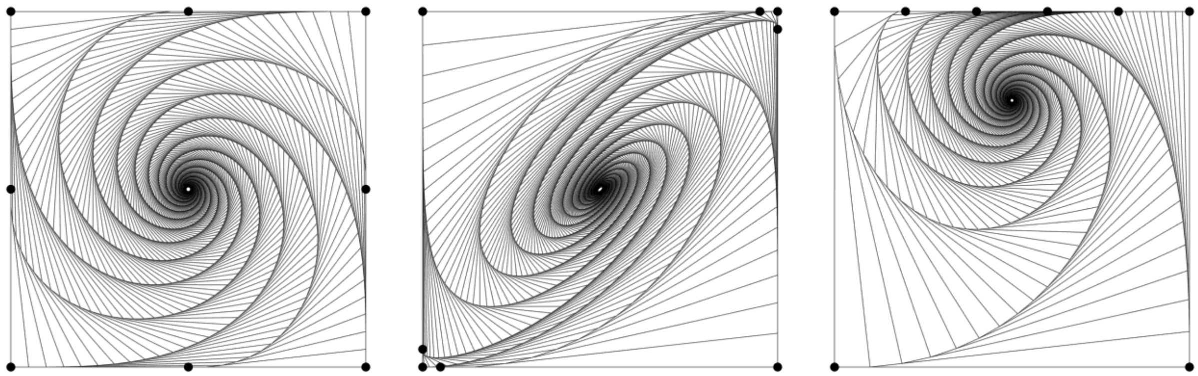
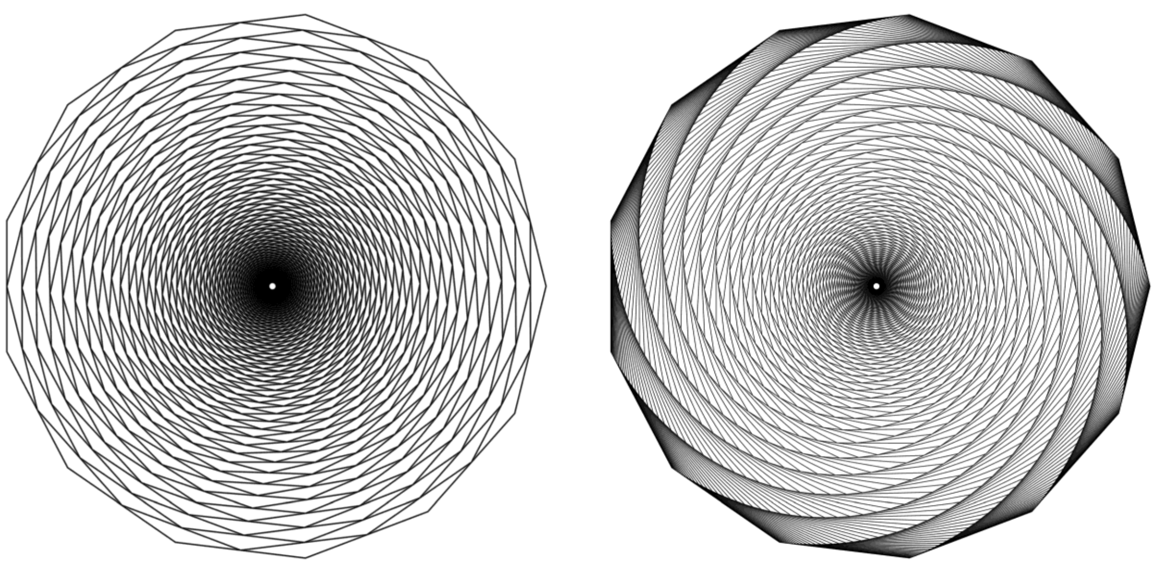


Figure 8: *Effects of locations of additional points on final whirls.*

As the number of points is increased, the individual curves get lost and larger patterns emerge. Setting the d values individually for each corner is less practical; another approach is to have d the same for each point, but vary with the iteration number, say from relatively low at the beginning and increasing as the iterations continue. In Figure 9 both panels show cases where the base is 13 points equally spaced around a circle and d is the same for each point. On the left, d increases linearly from 0.5 at the outside (first iteration) to 1 at the center. On the right, d varies sinusoidally with the iteration count, between 0 and 1.



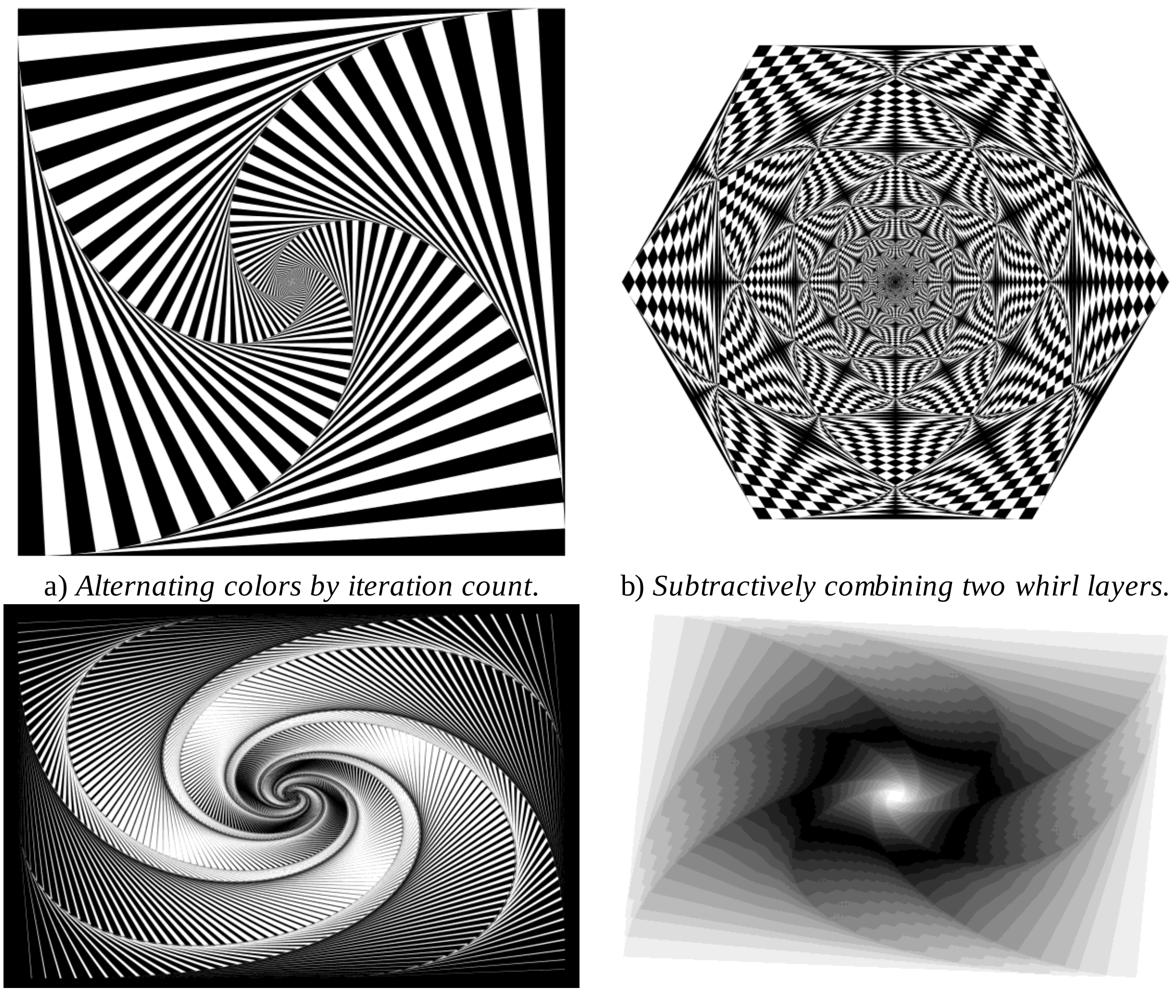
a) *Linear increasing d from 0.5 to 1.* b) *Sinusoidally varying d between 0 and 1.*

Figure 9: *Varying the relative distance with the iteration count.*

Combining Whirls into Images

All the whirls shown thus far have been rendered by drawing the outlines of the polygons. This highlights the internal structure and the pursuit curves. When combining whirls from different base shapes into a final image, other rendering approaches may be used to add aesthetic interest. One simple approach is to color the points inside the base polygon according to the number of the last iteration when that point was inside a polygon.

In panel a) of Figure 10, the square whirl from Figure 2 is rendered with alternating black regions. If the count of the last iteration for a given point was odd, the point is black, and white for even counts. In panel b), two such hexagonal whirls are layered. One is black and white with a counter-clockwise direction and the other is white and black with a clockwise direction. They are subtractively combined; points where each layer had the same color are finally rendered black and where they have different colors, the final image color is white. In panel c), the image also uses two layered whirls. In this case, the directions are the same, but the values of d vary slightly. Here, the subtractive combining serves to emphasize the boundaries of the regions, where the iteration counts differ. Such techniques can produce "op art" style images, reminiscent of the works of Bridget Riley [3] and Victor Varasely [4]. Finally, panel d) uses the same basic mechanics as in panel c), but employs a color palette with a smooth gradient from white to black. This changes the feel of the image from a high-contrast focus on the pieces to a more holistic consideration of the entire whirl.

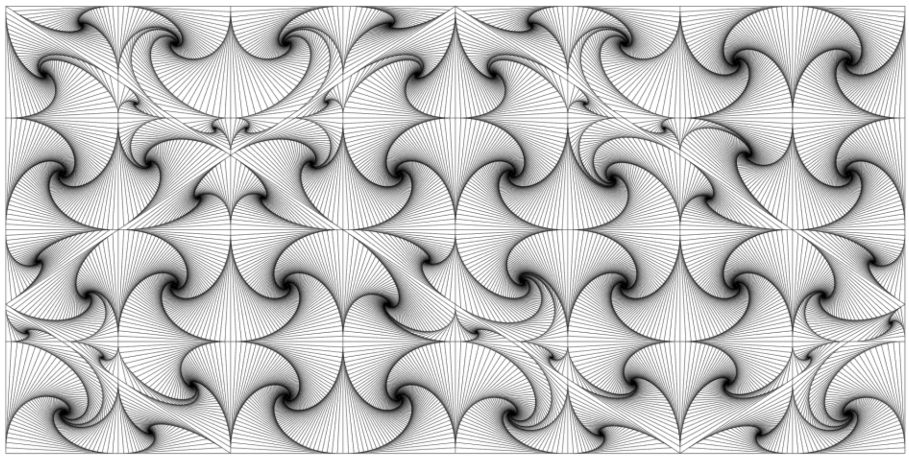


a) *Alternating colors by iteration count.* b) *Subtractively combining two whirl layers.*

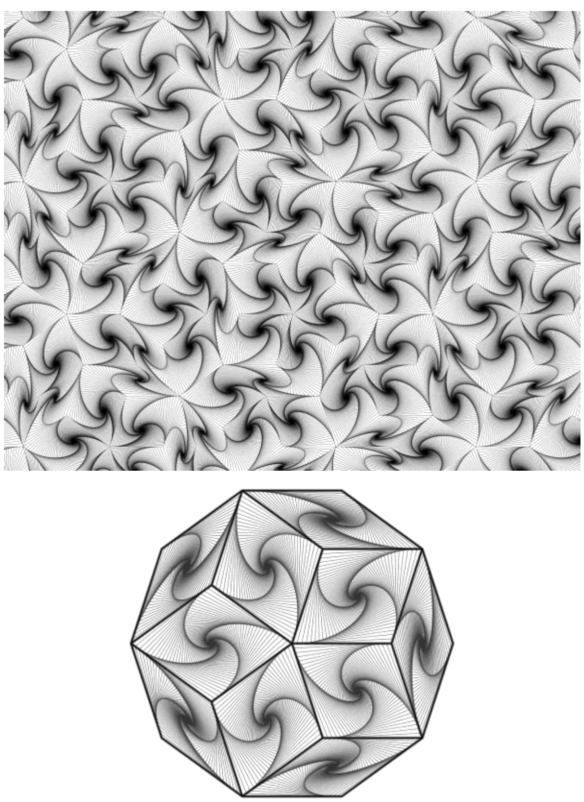
c) *"Op art" effect from layered whirls.* d) *Use of a smooth color gradient.*

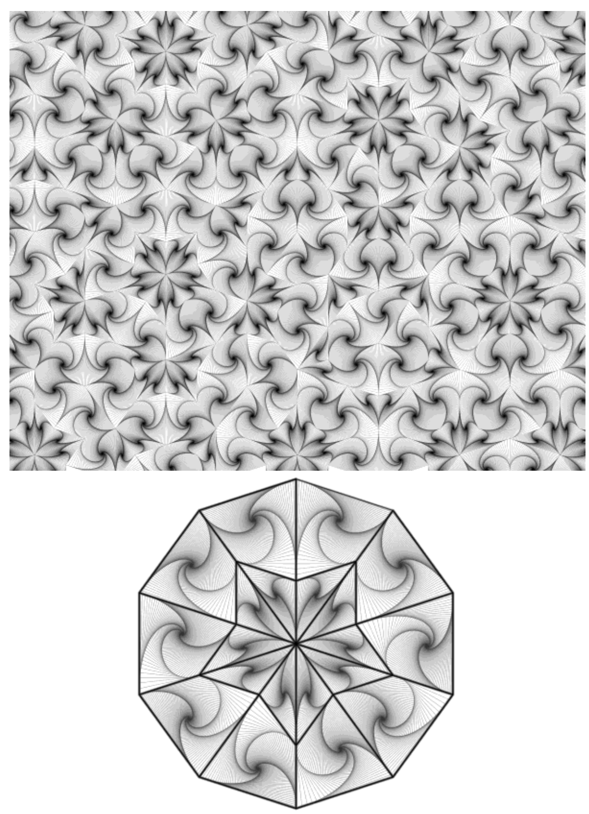
Figure 10: *Examples of alternative rendering methods.*

The primary motivation of this work to was to use pursuit curves to render polygonal tilings. If the parameters are chosen carefully, whirls can be used effectively with tilings, as the curves flow off of one tile into the next, seemingly seamlessly across the image. Figure 11 shows three examples by the author [5] using the outline technique. "Penrose Pursuit" (panel a)) and "Penrose Pursuit 2" (panel b)) use Penrose tiles [6] for the base polygons. These were chosen because they allow for aperiodic tilings, for greater visual interest. Below the images are small zooms showing how the tiles were used. In "Penrose Pursuit 2," the concave darts were each split into two triangle tiles, to avoid the issues described above.



a) *"Four-Corner Bank Shot," by the author.*

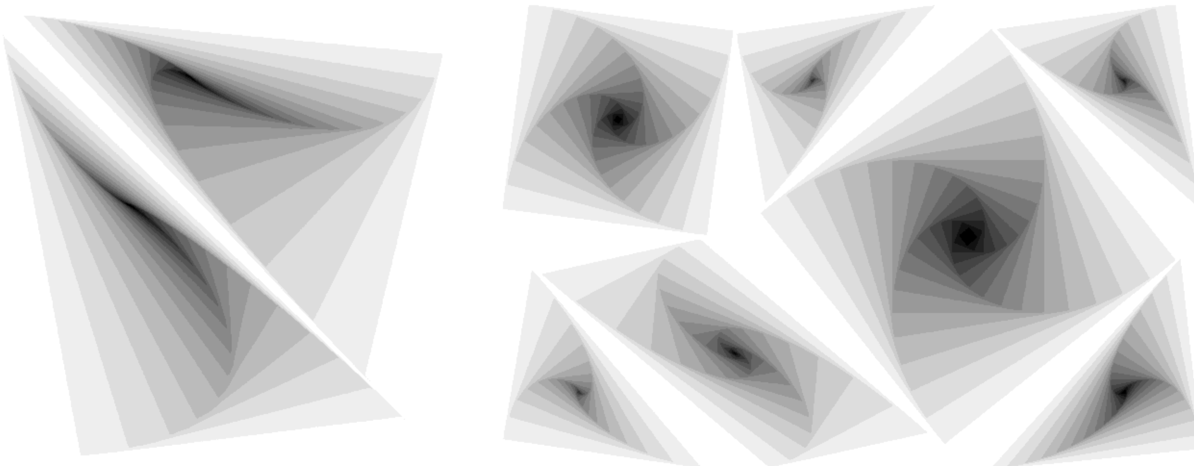




b) *"Penrose Pursuit," by the author.* c) *"Penrose Pursuit 2," by the author.*

Figure 11: *Rendering tilings using whirl outlines.*

Finally, Figure 12 shows two examples of tilings using the iteration-count method of coloring whirls. The subtlety of the gradient effectively hides the underlying tile shapes (two triangles in "She Sits" and seven squares and triangles in "Wormhole 1").



a) *"She Sits," by the author.* b) *"Wormhole 1," by the author.*

Figure 12: *Rendering tilings using whirls with iteration-count coloring.*

References

[1] Weisstein, Eric W. "Mice Problem." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/MiceProblem.html (as of February. 1, 2015).

[2] Weisstein, Eric W. "Pursuit Curve." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/PursuitCurve.html (as of February 1, 2015).

[3] http://www.op-art.co.uk/op-art-gallery/bridget-riley (as of February 1, 2015).

[4] http://www.vasarely.com/ (as of February 1, 2015).

[5] http://www.kerrymitchellart.com (as of February 1, 2015).

[6] Weisstein, Eric W. "Penrose Tiles." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/PenroseTiles.html (as of February 1, 2015).