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The Golden Spiral: The Genesis of a Misunderstanding

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Abstract

Among all of the unsupported assertions about the golden ratio one of the most puzzling is that the golden rectangle determines the proportions of the *Nautilus* shell, because it is demonstrably false. Given the flood of misinformation to be found on the internet it is probably inevitable that different versions of the idea are current, and some of them are not so obviously wrong, but it is surprising that it could have arisen in the first place. It is possible to trace its origins back into the nineteenth century, encountering some interesting methods of approximate construction along the way, but the mistake seems to be surprisingly modern.

Introduction

The debate about the occurrence of the golden ratio in art, design and nature is unlikely to be resolved any time soon. In some cases its presence is undisputed (for example in the work of artists who have stated that they have included it deliberately, any design based on the regular pentagon, and so on), but the list of controversial examples is rather longer, and continues to grow, apparently without limit. There is probably little new that can be added that will change the views of any of the protagonists, but some assertions relating to the golden ratio and logarithmic spirals that continue to be repeated uncritically are demonstrably untrue. Even the most committed enthusiasts admit this, once they look at the evidence [1] (although, typically, they then shift their ground), making the persistence of these ideas particularly puzzling.

The statement, "The spiral shape of the Chambered Nautilus (*Nautilus pompilius*) grows larger by a proportion of Φ – the Divine Proportion" is sadly only to be expected in a popular celebration of the golden ratio, [2] but similar assertions in educational books [3] by normally careful authors [4] are worrying. Even stranger is a Figure [5] bearing the caption, "A shell superimposed over a golden spiral over an acute golden triangle", with no further comment, when the spiral (Figure 6) does not match that of the *Nautilus* with which it is compared.

These anomalies become more understandable in the context of a few influential books from the early twentieth century, and their story provides an instructive episode in the history of ideas. The key individual is Theodore Cook, whose book, *Spirals in Nature and Art*, 1903, represents a watershed. Before its publication natural forms were certainly used by designers [6], but interest in mathematical aspects of both architectural and natural morphology was limited mainly to academics. Afterwards the "course of speculative generalisation which a more restricted method of study might have prevented" referred to by E.Ray Lankester in its preface was the beginning of a stream that has continued to grow until our own time. The book does not mention the golden ratio or Fibonacci numbers (even though phyllotaxis is briefly considered), and popular interest in the golden ratio probably did not begin, at least in the English-speaking world, until its appearance in Cook's *The Curves of Life* a decade later, even though Zeising [7] had promoted the golden section half a century earlier.

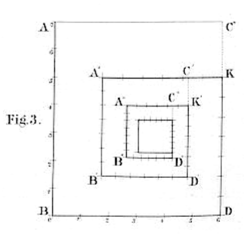
The Ionic Volute

Proportion has always been an important consideration in architecture. The Roman architect Vitruvius begins in his first book, second chapter (Of Those Things On Which Architecture Depends), with the statement, "Architecture depends on fitness ...", which he goes on to define as, "the adjustment of size of the several parts to their several uses, and requires due regard to the general proportions of the fabric". [8] When he goes into details (in books III and IV) his proportions are always whole-number ratios. Later writers followed Vitruvius, also recognising that slight deliberate variations had been introduced into classical architecture to compensate for optical distortion. J. Pennethorne took a particular interest in these corrections, and he was soon joined by F.C. Penrose, who made a comprehensive series of detailed measurements of classical ruins, and between them they published an important series of papers and books.

Pennethorne [9] includes a detailed consideration of the Ionic volute, and follows earlier authors by devising a method of construction that establishes the proportions between key dimensions of the capital and the rest of its column. He comments (p. 135)

> The idea of the spiral line, as that of other curved lines, is suggested by many forms that we find in nature: it may also be derived from the cone, by conceiving a continuous line winding round it from the base to the vertex. In the ancient geometry we have the spiral of Archimedes, and some of the spiral lines of the Greek ornaments will, perhaps, be found to be true examples of this form of curve; but in the case of the Ionic capitals, it is clearly stated by Vitruvius that the spiral lines are composed of circles, and the tracings taken from existing Greek volutes show clearly that this is the case.

He begins by constructing a spiral of straight lines based on a series of rectangles (Figure 1) that are connected by whole-number ratios, related to those of the complete column, and then needs to find the centres of the circular arcs. He describes how this caused him some difficulty until John Robinson, who drew the plates for the book, found the method that he describes. It requires a linear progression in the sides of the first three rectangles (Figure 2), so the first three quadrants approximate to an Archimedian spiral. The construction is repeated for each of the nested rectangles, which are related by a fixed ratio, so his volute, which has been specifically designed to be aesthetically acceptable, amounts to a logarithmic spiral approximated by arcs of (approximate) Archimedian spirals.



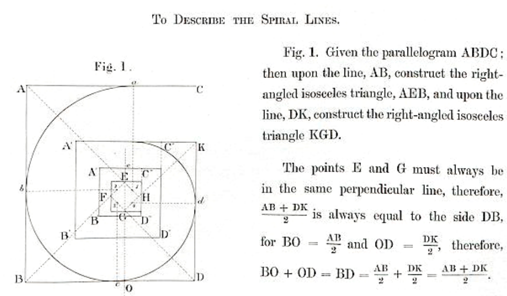

To Describe the Spiral Lines.

Fig. 1. Given the parallelogram ABDC; then upon the line, AB, construct the right-angled isosceles triangle, AEB, and upon the line, DK, construct the right-angled isosceles triangle KGD.

The points E and G must always be in the same perpendicular line, therefore, $\frac{AB + DK}{2}$ is always equal to the side DB, for BO $= \frac{AB}{2}$ and OD $= \frac{DK}{2}$, therefore, BO + OD = BD $= \frac{AB}{2} + \frac{DK}{2} = \frac{AB + DK}{2}$.

Figure 1: *Pennethorne's rectangular spiral.* **Figure 2:** *His spiral constructed from circular arcs.*

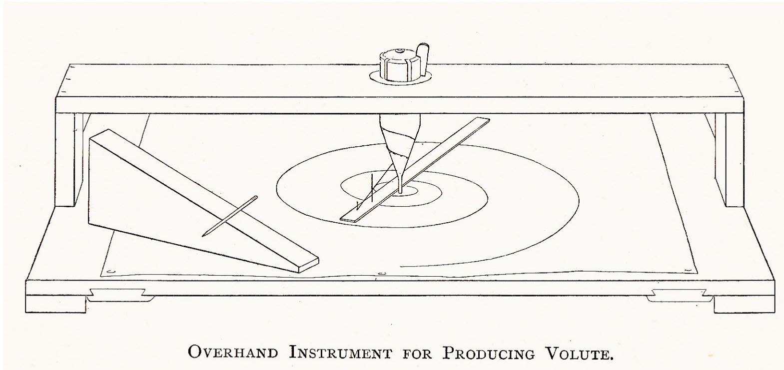


Figure 3: *Volute construction from a cone.*

The construction using a cone (Figure 3) [10] will generate an approximation to the involute of an Archimedian spiral because the error introduced as the generating radius deviates from the horizontal is small. Banister Fletcher (Figure 4) [11] refined the method by using a whelk shell instead of a cone. Fletcher does not discuss his innovation, but its most obvious advantage is that the cotton thread is fixed in the groove around the shell, so it will not slip. In addition, since the groove is a helix corresponding to a logarithmic spiral, and the spiral is an approximation to its involute, which is itself a logarithmic spiral.

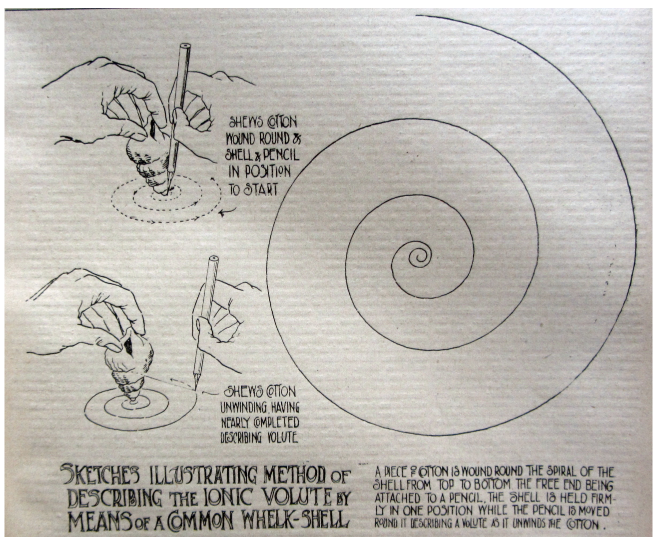


Figure 4: *Banister Fletcher's method of construction.*

Spirals in Nature and Art [12]

Theodore Andrea Cook was a journalist and sportsman with a wide range of interests. [13] By the time *Spirals in Nature and Art* was published in 1903 he had already published books about French c*hâteaux*, [14,15] and his primary aim was to establish Leonardo da Vinci as the designer of the *Escalier à Jour* in the *Château de Blois* at Touraine. He had already made this proposal in his earlier work, as well as noting its similarity to the structure of a shell. *Spirals* expands on this idea by considering a wide range of spirals (actually mainly helices) in nature and art as the title indicates. To quote again from the preface, “The investigation of spiral forms in Nature is itself a department of scientific enquiry that is still very incomplete, and I believe I am correct in thinking that neither in mathematics nor in architecture has the last word yet been said upon this interesting subject.” The influence of this book, and Cook's job, first at the *Daily Telegraph* and later (1910-28) as the editor of *The Field*, was to have a profound effect on the popular interest on spirals, and, more generally, on morphology in science and art.

Spirals, as opposed to helices, are touched on on pages 70-72, in connection with Church's explanation of phyllotaxis, [16] and again on pages 122-6 in his chapter titled, “The Nautilus”, where he cites John Leslie [17] and H.Moseley [18] on shells and logarithmic spirals. The book finishes (pp. 180-190) with a more general discussion on, “... the subtle and elusive causes of beauty ...” with particular reference to the logarithmic spiral and the possibility that it might be, “... fundamental for all forms of growth ...”. Cook's major point is that irregular and subtle variation is essential to beauty, and he cites Fletcher's method for drawing an Ionic volute as a further example, since it uses an organic template, a whelk shell, which is necessarily irregular.

The Curves of Life [19]

In Cook's next book on spirals, *The Curves of Life*, 1914, he expands considerably on his earlier work, and draws extensively on contributions from other people, especially many that had been published in *The Field*, which he had edited since 1910. In his preface (almost certainly written last) he states, “The Formula for Growth now suggested in this book is called the φ spiral, or Spiral of Pheidias, a new mathematical conception worked out from an ancient principle by Mr. Mark Barr and Mr. William Schooling.” What he means by this is made clear in his final chapter (p. 419 *et seq.*). After he had published Church's theory of phyllotaxis in *The Field*, William Schooling had written to him with an explanation of the golden section, including a suggestion by Mark Barr that it be called the φ proportion, “... partly because it has a familiar sound to those who wrestle constantly with π [indicating the English pronunciation, fie, rather than the American, fee] … and partly because it is the first letter of the name Pheidias ...”. He goes on to consider what he calls The Pheidias Spiral, where, “... the radii vectores [measured from the pole to successive crossings of the spiral] of a logarithmic spiral are in φ proportion, the result is not only a spiral of singular pleasing character, but there is a further feature that on any radius the sum of the distances between two successive curves of the spiral equals the distance along the same radius to the succeeding curve (see Fig. 389 [Figure 5]). Such a φ spiral bears a close resemblance to the spiral … produced by unwinding a tape from a shell.”

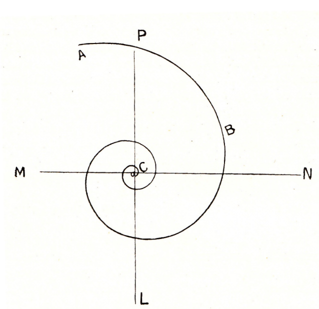


FIG. 385.—LOGARITHMIC SPIRAL WITH RADII.

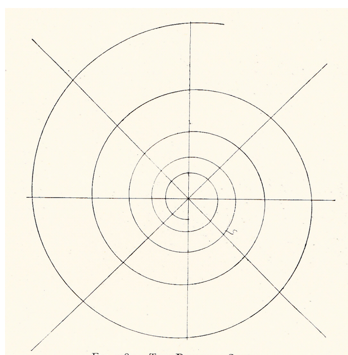


FIG. 389.—THE PHEIDIAS SPIRAL.

Figure 5: *Cook’s generic example (left) compared with his Pheidias spiral (right).*

It is clear from what follows that Cook has been convinced, probably by Schooling, that the golden ratio underlies many works of art, but it is equally clear that he does not see it in the Nautilus shell. He comments, "... it will not imply that the artist had any preconceived idea of using the φ proportions in his composition, any more than the Nautilus had any conscious plan of developing a certain spiral in its shell." This is his analogy: proportions in beautiful art are to φ as the shell of Nautilus is to a logarithmic spiral, but his figure of what he calls the φ spiral is nothing like a Nautilus.

Unfortunately Schooling's account in the appendix begins, "The chief interest in relation to Mr. Cook's inquiry into the principles of growth and beauty is, on the one hand, its connection with the Fibonacci series and phyllotaxis, and, on the other, with the φ spiral, an illustration of which was published in the *Field*, and which throws some light on numerous spiral formations in Nature and art (see Fig. 385)." Cook's φ spiral is illustrated in his Fig. 389, not Fig. 385, which is a generic logarithmic spiral. Schooling may have been the first person to misunderstand the relation between φ and logarithmic spirals, but that is unlikely since he was an actuary with a mathematical background. It is more likely that some of the confusion, which Huntley [20] later compounds by devoting the last chapter of his book on the golden ratio to logarithmic spirals, as well as using the *Nautilus* shell as a frontispiece, is the result of a misprint.

On Growth and Form [21]

D'Arcy Thompson devotes just over 150 pages of nearly 800 in *On Growth and Form*, published three years after *Spirals*, to the same material as Cook, but from a different point of view. At a time when natural selection was invoked to explain any biological phenomenon he argued that there are some aspects morphology that result directly from purely physical causes. In particular, logarithmic spirals and helices are an inevitable result of the mechanisms of growth: it occurs by the steady accretion of material (horn, shell etc.) onto what is already there, and it is likely that Schooling, who was primarily a financier, was interested in growth by the obvious parallel with compound interest. Thompson considers many examples of such growth, including the golden rectangle and the 72° isosceles triangle, although in the first edition he constructs a spiral only for the triangle (Figure 6). He repeats Moseley's analysis of shells, beginning with *Nautilus*, and notes that the widths of adjacent whorls are in the ratio of approximately 3:1.

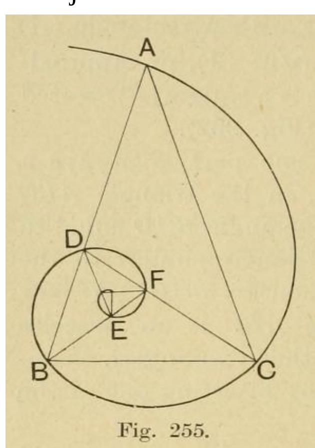


Figure 6: *D'Arcy Thompson's logarithmic spiral around the 72º isosceles triangle.*

Thompson is very critical of Church and Cook when he considers phyllotaxis, particularly their implicit vitalism, "Mr Church sees in phyllotaxis an organic mystery, a something for which we are unable to suggest any precise cause ...", and he identifies Chauncey Wright as the first to give a mathematical explanation of the significance of Fibonacci numbers, quoting his 1871 paper. [22] He is clearly unaware of Wright's 1859 paper [23] or his publication in *Gould's Astronomical Journal* in 1856.

There are interesting contrasts in the different inferences drawn from the evidence of Fibonacci numbers in phyllotaxis. Wright is best known as an empiricist philosopher who embraced Darwinism and was a vigorous critic of metaphysics and the German school of *Naturphilosophie*. For him phyllotaxis had arisen by natural selection as the most efficient arrangement of leaves. Thompson argued that it was not necessary to invoke natural selection to explain all aspects of morphology, and phyllotaxis is simply the result of leaves originating where there is most space. Church is in the tradition of *Naturphilosophie*, along with Cook, and Thompson attacks him on those grounds.

The Golden Rectangle

Although Fechner's researches [24] were based on rectangles, the golden rectangle is not mentioned by Cook (who cites Fechner in the last chapter of *Curves*). Thompson includes it as an example of gnomonic growth, but it is not until the 1942 edition of *On Growth and Form* that he shows a logarithmic spiral passing through its vertices, analogous to his golden triangle construction (Figure 6).

Jay Hambidge first presented his analysis of the Parthenon to the Society for the Promotion of Hellenic Study (London) in 1902 ([25] p. 138), and in a preface (dated 1 Dec. 1911) Colman states that he, "… was among the first students to disclose the true symmetry of shells". [26] He begins *Dynamic Symmetry, The Greek Vase* with a brief description of logarithmic spirals constructed by using right-angled triangles, but continues, "As far as design is concerned, we may now dispense with the curve of the spiral." He proceeds to construct special rectangles, only one of which is the golden rectangle (which he calls the whirling squares rectangle), which become the bases of his analyses for the rest of the book. This is an interesting reversal of the usual sequence, such as that described by Pennethorne, where a spiral is derived from a series of rectangles.

In the literature, beginning with Thompson, the logarithmic spiral relating to the gnomonic growth of a polygon is generally illustrated by constructing it through the vertices (as in Figure 6), and this practice has continued up to the present. [27] A web search for images of the golden spiral produces almost no examples of spirals through the vertices of the rectangle. Almost all are tangential to the sides (compare with Figure 2), suggesting that they all illustrate the approximate construction from circular quadrants. Often the pole of the spiral is determined by intersecting diagonals (following Hambidge), which are radii of the spiral rotated so as to pass through the vertices.

Nautilus pompilius

Among all the species of molluscs there will certainly be some with shells that match, for example, Cook's Pheidias spiral, so why should *Nautilus pompilius* be cited as the example of the "golden spiral"? Cook provides a clue in his chapter on flat spirals in shells in *Curves of Life*. He begins, "Among all the flat spirals shown in shells it has long been recognised that *Nautilus pompilius* exhibits the most beautiful of all, and one so closely akin to mathematical curves that Sir John Leslie … wrote: 'This spiral exactly resembles the general form and elegant septa of the nautilus'." If the nautilus is the most beautiful, and you believe that φ is the measure of beauty, then there must be a connection. The easiest logarithmic spiral to construct (approximately, from circular quadrants) is the one that grows by a factor of φ every quarter turn, from the whirling squares of the golden rectangle, so the two must be the same. Clearly this is just lazy thinking, and the construction lines of the rectangles are sufficiently distracting to make the difference between the two spirals less visually obvious, and there seems little reason to question the assertion.

Summary

All of the pieces of this story originated during the nineteenth century or earlier: methods to draw spirals for Ionic volutes have been described for hundreds of years; detailed measurements of important classical monuments were made by Penrose, and used by him and Pennethorne to investigate aspects of Greek architecture throughout the century, and later by Hambidge; the shapes of shells were known to be logarithmic spirals by the early nineteenth century; a materialistic explanation of Fibonacci numbers in spiral phyllotaxis was proposed in the middle of the century; Zeising's ideas about the golden section were published just a few years earlier, and Fechner developed them about twenty years later. At the beginning of the twentieth century Theodore Cook, starting from his conviction that a particular staircase in a French *chateau* was designed by Leornado da Vinci, wrote a book that stimulated interest in spirals, presented the logarithmic spiral to a popular audience, and publicised Church's theory of phyllotaxis. This stimulated correspondents, in particular William Schooling, to communicate ideas about the golden section, probably derived from Zeising and Fechner, which were quoted in Cook's second book. Meanwhile Jay Hambidge analysed the Parthenon, based on Penrose's measurements, and developed his theory of Dynamic Proportion, which he related to logarithmic spirals.

D'arcy Thompson published On Growth and Form in 1917, taking a much more rigorous approach, and this influential book disseminated ideas about morphology even more widely, probably making them more respectable in the process. A large section of the work deals with logarithmic spirals, including phyllotaxis, and mention is made of the golden section in the chapter on shells. Reference is made to the previous work of Church and Cook, as well as Zeising (in passing), although not without serious criticism.

Although none of these publications makes an explicit link between logarithmic spirals in general and the specific value of the golden section, there is a very clear association of ideas, reinforced by what seems to be a misprint in Schooling's contribution to the appendix of *Curves of Life*. Since the shell of *Nautilus pompilius* is one of the best known examples of a logarithmic spiral in nature it is perhaps not surprising that it came to be associated with the golden rectangle, although I am still unable to answer a question that John Sharp asked me several years ago: where is it first stated explicitly that the parameter of the Nautilus spiral derives from the golden rectangle? The earliest examples I can find date from the 1980s.

Once an idea enters popular consciousness it can maintain a tenacious existence, and a recognition that it is wrong can taken as a challenge: if it is believed that the golden ratio is everywhere, and it is said to underlie the proportions of the shell of *Nautilus*, how might this might be true? This seems to be what motivates the argument in [1], where, after demonstrating that the usual assertion is wrong, evidence is presented to show that the shell grows by a factor of φ every half-turn.

Genuine unexpected connections, such as the importance of Fibonacci numbers in phyllotaxis, can stimulate an interest in mathematics and motivate further study, but education should also develop habits of critical thought, (compare with the examples of [2], [3], [4] and [5]). The history of the *Nautilus* and the logarithmic spiral illustrates how reasonable ideas can give rise to misunderstandings, which are then repeated uncritically, leading ultimately to the acceptance of demonstrably false assertions. It would form an instructive case-study to use along with more usual motivational art and mathematics topics, particularly the golden ratio and Fibonacci numbers.

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