The Platonic Solids: a Three-Dimensional Textbook

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Abstract

The Platonic Solids: a Three-Dimensional Textbook is an exhibit in the 2015 Bridges conference that includes geometric sculptures, depicting the geometric relationships between the five Platonic solids, as well as extensive text explaining and developing those geometric ideas. The Platonic solids serve as a very accessible vehicle for introducing ideas from projective geometry and group theory. It is fitting to depict beautiful mathematical ideas with beautifully crafted models. It is hoped that others will in the future be inspired to make "three-dimensional textbooks" for other topics in geometry.

Introduction

The Platonic Solids: a Three-Dimensional Textbook is an exhibit in the 2015 Bridges conference. The richness of ideas found in the Platonic Solids gives a wonderful example of the beauty to be found in mathematical thought. A geometry textbook typically develops the ideas with text and two-dimensional drawings. However, three-dimensional geometry is most effectively presented in three dimensions. So, this "textbook" gives a systematic development of geometric ideas, but does so through a series of more than thirty 3D sculptures with explanatory signs. The beautiful sculptures give a fitting expression to the beauty of the mathematical ideas -- a bridge between sculpture and mathematics.

The intended audience for the exhibit is anyone who has studied a little high school algebra and geometry, but someone with no mathematical background should find it quite accessible, while those with a sophisticated mathematical background will likely find there some delightful surprises. This paper summarizes the content of the exhibit; it assumes that the reader has some familiarity with some of the polyhedral forms ([3],[4],[6]) and basic projective geometry ([2]), perhaps from having gone through the exhibit. It is hoped that the exhibit and this paper will encourage others to use the Platonic solids as a way to introduce in a very concrete way some abstract mathematical concepts, and perhaps it will also encourage others to create 3D textbooks for other topics in mathematics.

Imagination in Mathematics and Art

Mathematics is an art form. It is an art form in which the medium is pure thought. It is unlike other art forms in that other forms find their expression in physical material. Perhaps we should describe Bridges as an organization dedicated to making connections between mathematics and *other* arts. In any case, we should not be surprised that we so readily find many connections between mathematics and other arts.

Art stimulates inner activity. The sculptures in this collection do so by calling upon the viewer to inwardly form images and visualize forms in movement, to see in the mind what is suggested by, but is not physically present in, the pieces. There is a multitude of dynamic movement both within each sculpture and between the sculptures. That interaction with the viewer engages one in the choreography of these movements.

The viewer will find it interesting to look at the shadows cast by the pieces. Also, if one moves about, viewing the pieces from different perspectives, various lines and points will suddenly line up, giving quite stunning views; try this also with one eye closed.

The Convex Regular Solids

In ancient Greece, it was already proved by Euclid that the five figures, known as the Platonic solids, exist and are the only convex regular polyhedra. Also Plato discussed their symbolic significance in *Timaeus* [4]. A table, showing the number of vertices, edges, and faces of each of them, is displayed in the exhibit. It reveals a pairing of the octahedron and cube with the role of vertices and faces interchanged, and likewise for the icosahedron and dodecahedron. We can understand this better in the context of projective geometry.

The Infinitely Distant and Duality in Projective Geometry

A plane, P, intersects a second plane, Q, in a line, q. As plane P moves about in space, the line q moves about on plane Q. As plane P comes close to being parallel to plane Q, line q moves far away. At the moment when P becomes what Euclid would have called "parallel" to Q, Euclidean geometry says that there is no longer a line of intersection; q disappears. However in projective geometry, one includes an infinitely distant line on every plane; so the line q is still there, only infinitely distant. As a result there is a principle of duality in projective geometry (which is lacking in Euclidean geometry); any two points have a common line and any two planes have a common line. In three dimensional projective geometry, interchanging the words "point" and "plane" converts any true statement to another true statement.

Polarity in Projective Geometry

The principle of duality interchanges the words "point" and "plane" in any general statement, but it doesn't associate to a point any particular plane. That's done by a polarity. A polarity pairs each point in space to a plane in such a way that incidence is preserved; if a point lies on a plane, the polar plane and point lie on each other. It follows that as a point moves along a line, the polar plane must rotate around another line; thus each line has a polar line. An example of a polarity associates to each point with homogeneous coordinates, (a, b, c, d), the plane with equation $ax + by + cz - dw = 0$. Then each point on the unit sphere has for its polar the plane tangent to the unit sphere at that point. The polar of the sphere's center point is the plane at infinity. As a point moves toward the center of the unit sphere, the polar plane moves out to infinity and conversely. Once a polarity is chosen, every polyhedron has a polar polyhedron. From now on we will use "polarity" to mean this particular polarity, and we will call the unit sphere "the sphere of polarity".

Polars of the Platonic Solids

Using this polarity, one sees that the cube and octahedron are polar to one another. The vertices of the cube line up radially with the centers of the faces of the octahedron and vice-versa, while the midpoints of the edges of the cube line up radially with the midpoints of the edges of the octahedron. Making the cube smaller relative to the sphere of polarity makes the polar octahedron larger and vice-versa. By varying the relative sizes of the cube and polar octahedron, one finds many interesting relations. In the two models displayed in the exhibit, the aluminum cube and brass octahedron have been chosen with relative sizes so that they can be suspended from one another with strings, the strings forming a dodecahedron in one model and an icosahedron in the other. Some of the vertices of the string dodecahedron lie on vertices of the cube, while its other vertices lie on edges of the octahedron; moreover, its faces all lie on edges of the cube. The dual relations hold for the string icosahedron in the other model. I have not seen these models elsewhere in the literature or in other exhibits.

Likewise the dodecahedron and icosahedron are polar to one another. Shown in the exhibit is a progression of five models, entitled "Expansion and Contraction in Polarity." The sphere of polarity's size is kept constant. The brass icosahedron becomes progressively smaller, resulting in the polar aluminum dodecahedron becoming progressively larger. In the middle model, the two polyhedra cross at the midpoints of their edges. The ratio, (dodeca diam)/(icosa diam), is multiplied by the golden mean as one moves from each model to the next. In this way, many interesting geometric relations can be seen in each of the stages of the progression. Moreover, it is a challenging but interesting exercise, while viewing the models, to imagine a continuous movement from each model to the next one.

So, the cube and octahedron are polar to one another and likewise the dodecahedron and icosahedron. What about the tetrahedron? Should its polar not give a sixth Platonic solid? No, Euclid was correct; there are only five. The tetrahedron is special; it is polar to itself, as shown here in one model.

Rotation Groups

There is a remarkable theorem, due to Euler, which tells us that any orientation preserving rigid movement of a sphere is a rotation about an axis. It follows that such a symmetry of any finite figure is likewise a rotation about an axis, and that all of the rotations of a finite figure form a group. Some figures, such as an n-sided prism, have rotation groups that are *reducible*; although the prism is three dimensional, its rotation group is just a combination of symmetries of one and two-dimensional figures; the symmetry is not really three dimensional. There is a second remarkable theorem, which tells us that any irreducible group of rotations of a finite figure has the same rotational symmetries as one of the Platonic solids. ([1], [3], [7]) So, there are just three finite irreducible rotation groups: the tetrahedral, octahedral, and icosahedral. The ancient geometers found that Platonic solids express something fundamental about the nature of space in that they are the only convex regular solids. In modern times we again see them to be fundamental as representing the only finite rotational three-dimensional symmetry.

There are eight models here, showing all of the axes of rotation for each of the three Platonic solid groups. The number of rotations of each group is tallied, giving 12, 24, and 60, for the tetrahedral, octahedral, and icosahedral groups, respectively. One asks, "Why are they all multiples of 12?" This will be answered in what follows.

The Tetrahedron and Subgroups

The tetrahedron is very special. It has fewer vertices, fewer edges, and fewer faces than any other polyhedron. Moreover, the regular tetrahedron can be oriented in relation to each of the other Platonic solids so that all of its rotations are also rotations of the other Platonic solid. In other words, the tetrahedral group is a subgroup of the octahedral group and also a subgroup of the icosahedral group.

Figure 1: *Two tetrahedra (one brass, one aluminum) and a cube and octahedron in string. Photo: Jason Pogacnik*

Figure 1 shows that the tetrahedron can be inscribed, vertex on vertex, in a cube, and dually that it can be circumscribed, face on face, about an octahedron. Of the octahedron's 24 rotations, 12 are also rotations of each of the two circumscribing tetrahedra. The other 12 rotations of the octahedron interchange the brass and aluminum tetrahedra. A model in the exhibit, not pictured in this paper, comprises five tetrahedra in brass tubes, suspended from one another by wires that form a circumscribed dodecahedron; it also has an aluminum icosahedron suspended inside, showing the intersection of the five tetrahedra. A second model, made by Steve Morse in cardboard, shows the same five tetrahedra; here one can easily imagine the circumscribing dodecahedron. The theorem of Lagrange, that the order of a subgroup divides the order of a finite group, is proved by noting that the equal-sized cosets partition the group. This model of five tetrahedra gives an example in which the cosets can be pictured very concretely. In fact, consider one of those tetrahedra, say the red one. Of the dodecahedron's 60 rotations, 12 of them are also rotations of the red tetrahedron. Thus, the tetrahedral group is a subgroup of the icosahedral group. Moreover, of those 60, 12 of them (one coset) will move the red tetrahedron to, say the blue tetrahedron, and 12 more to each of the other colors. So, we see very concretely why the order of the icosahedral group is five times the order of the tetrahedral group. Moreover, the rotation groups of the five tetrahedra are subgroups of the icosahedral group and illustrate well the idea of conjugate subgroups.

Next in the exhibit is a model, showing all five Platonic solids in one figure, with the dodecahedron and cube sharing vertices with the tetrahedron, while the octahedron and icosahedron share face planes with the tetrahedron. Take any two Platonic solids and orient them so that four axes of order 3 coincide; this can be done, since they both have a tetrahedral group among their rotations. Then by varying the relative sizes of the two solids, one finds many interesting and beautiful geometric relationships, some of which are shown in this exhibit.

Nested Sequences

In the series, "Expansion and Contraction in Polarity," the second model is a dodecahedron suspended inside an icosahedron, and the fifth in the series is an icosahedron suspended inside a dodecahedron. One can imagine combining these two to make an infinite sequence of dodecahedrons and icosahedrons, alternately suspended inside each other. Figure 2 below shows a brass dodecahedron on the outside with an aluminum icosahedron suspended inside it and finally another small dodecahedron on the very inside.

This model is accompanied in the exhibit (not shown in this paper) by its dual, which has a large and a small icosahedron and a dodecahedron in between.

In figure 2, the ratio of the large dodecahedron to the small one is the cube of the golden mean. There must be two other dodecahedra in between the small and large ones, with ratios to the small one of the golden mean and the golden mean squared, respectively, creating thereby a geometric progression. What would that look like? It turns out that if a cube is inscribed in a dodecahedron, edge on face, and another dodecahedron is inscribed in the cube, edge on face, then the two dodecahedra have ratio of the golden mean. Figure 3 below shows a sequence of four brass dodecahedra, nested with 3 black string cubes in between.

Figure 2: *Two dodecahedra with an icosahedron suspended between. Photo: Jason Pogacnik*

Figure 3: *A nested sequence of dodecahedra and cubes. Photo: Jason Pogacnik*

The Kepler-Poinsot Polyhedra

If one relaxes the requirement of convexity, then in addition to the five Platonic solids, four more regular solids exist, known as the Kepler-Poinsot polyhedra. These four appear in the exhibit, beautifully rendered in wood by Bob Rollings. They are accompanied by a model which is the dual of the one in figure 2 above. If one looks attentively, one can find within that model every one of the four Kepler-Poinsot polyhedra.

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