Real-World Tessellations

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Abstract

We present photographs of natural and synthetic tessellations taken over a period of several years in locations around the world. We give several examples that illustrate how and where tessellations arise in the real world. Line drawings of tessellations based on the photographs are included in some cases to make points about the characteristics of the tessellation.

Introduction

During my travels over the last several years, many of which included a Bridges Conference, I have photographed tessellations (tilings) that I have observed. Nearly 100 of these have been collected in a new book targeted at K-12 education [2]. These can broadly be categorized as natural or synthetic. Tessellations in nature are often not very regular in structure, and no real-world tessellation strictly meets the mathematical definition of a tessellation. Recall that a tessellation is a collection of shapes that fit together without overlaps or gaps to cover the infinite mathematical plane [5]. In a real-world tessellation, the lines or gaps between the tiles will always have finite width. The individual tiles in what is nominally a repeating tessellation will never be exactly like one another. No real-world tessellation can be infinite in extent, and no planar surface in the real world is perfectly flat. For the purposes of this paper, these irregularities will be largely ignored so that a brick wall, for example, can be described as a repeating tessellation of identical rectangles. Mathematically, one can also describe non-planar surfaces as being tessellated, and there are natural surfaces that are approximately cylindrical, spherical, etc. The bark of a tree trunk could, for example, can be modeled as a tessellation of a cylinder.

Beyond natural vs. synthetic, these tessellation photographs can be grouped in various ways. One is by where they were taken. Another is by the type of object shown. The geometry of the tessellation is another obvious way to group them. Yet another is the phenomenon responsible for the tessellation; e.g., tessellations created by a cracking process can be observed in synthetic and natural objects. Each way of looking at them adds a dimension to our understanding of the role of this particular type of mathematics in the world around us.

All of the photographs shown in this paper were taken by me, and all are visible to the naked eye. The microscopic world contains a whole range of tessellations, but those are not discussed here. In the following, tessellations are primarily categorized as natural or synthetic, but other categorizations will be commented on as well.

Tessellations in Nature

Nature can be thought of as the glue or the substrate that binds art and mathematics, and observing nature teaches us about both art and math. On the macroscopic scale, there are several categories that natural tessellations tend to fall in. One of the most common is a segmented covering of a surface, such as the scales covering a reptile. Figure 1 shows the scales on a lizard, where the scales on the hind leg approximate a regular tessellation of hexagons. Another type of barrier formed using a tessellation is a spider's web. Examples of synthetic tessellations of this sort include nets and chain-link fences.

Figure 1: *The scales of a lizard's leg can be modeled as a tessellation of hexagons. Photographed near Tucson, Arizona, USA.*

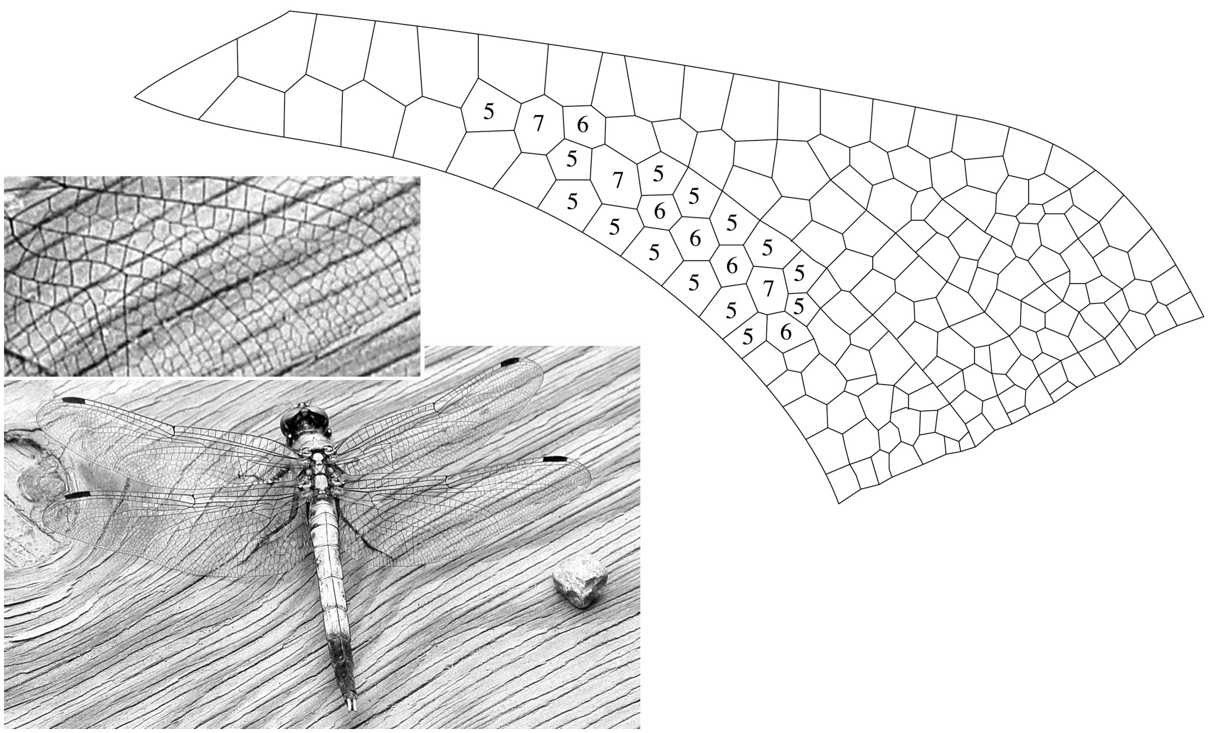

Figure 2: *A dragonfly's wing exhibits a complex and beautiful tessellation. On the line drawing of a portion of the zoomed-in region, each number indicates the number of edges on the containing tile. Photographed at Koishikawa Kōrakuen Garden, Tokyo, Japan.*

Divisions of surfaces in plants and animals can result from purposes other than a protective covering, such as distribution of fluid to all parts of a leaf through a network of veins. Another example is the veins on a dragonfly's wing (Figure 2). The long straight veins create rows of generally pentagonal tiles, with hexagonal tiles filling the space between them. However, when a new row of tiles is added as the wing broadens outward, a pentagonal tile is often seen, followed by a heptagonal tile, after which hexagonal tiles predominate. Another type of tessellation found in living creatures is a pigment pattern that could serve as a camouflage mechanism, as shown in Figure 3.

Many natural tessellations like the spots on a giraffe are reminiscent of Voronoi (Dirichlet) tessellations [5]. These are generated from a collection of points by drawing lines that are equidistant between two nearby points and perpendicular to the line joining them. Each resulting tile is the collection of points that is closer to the generating point it encloses than to any other generating point. At left in Figure 3, points are located by eye in the center of some of the giraffe's spots, and at right the Voronoi tessellation generated by these points is shown in black along with a drawing of the natural tessellation in gray. The Voronoi tessellation is seen to closely approximate the natural tessellation, which may have implications for biological models of giraffe pattern formation.

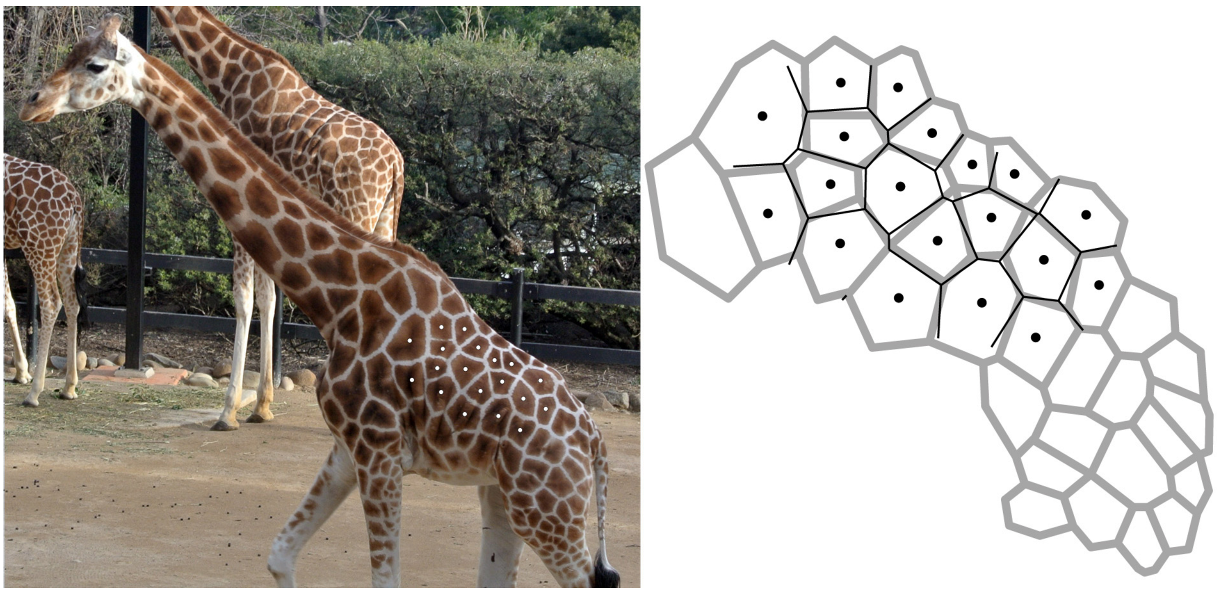

Figure 3: *The spots of a giraffe can be thought of as a tessellation. In the patch of tiles drawn from the side of the front giraffe, there are two quadrilaterals, 16 pentagons, 11 hexagons, and 6 heptagons. The dots were used to generate a Voronoi tessellation as described in the text. Photographed at the Taronga Zoo, Sydney, Australia.*

Tessellations resulting from cracking are often observed in inorganic materials. Cracking is generally a stress-relief mechanism, with cracks nucleating at defects and then propagating. This is commonly observed in dried mud (Figure 4, left), due to contraction of the mud as its water content decreases. It's also seen in the natural cracking of synthetic objects like a sheet of glass (Figure 4, right).

Tessellations in Synthetic Objects

The use of tessellations by people is both utilitarian and decorative. People of virtually all cultures have used geometric designs for decoration of cloth, pots, floors, masks, and even their own bodies [8]. In these applications, tessellations are used to make surfaces more beautiful or more interesting, though utilitarian aspects can be integral to the design.

Many of the tessellations employed by people are periodic, allowing them to be classified according to the 17 plane symmetry, or Wallpaper, groups. The wallpaper groups of the tessellations in Figures 5-8 are specified in the following paragraphs.

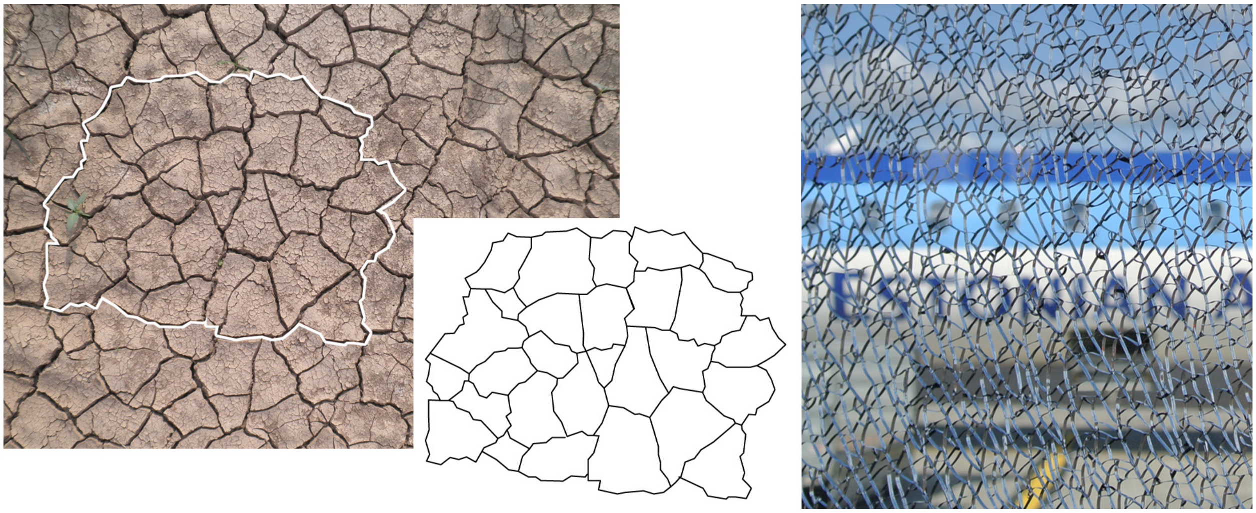

Figure 4: *The cracks in dried mud create a tessellation (left). Modeling this with a line drawing reveals little regularity beyond the size of the tiles. Photographed at the edge of a sunflower field near Dombóvár, Hungary. Networks of cracks are also observed in synthetic materials such as plate glass (right). Photographed in Oslo, Norway.*

In practical applications, it is often the case that dividing up a large structure into smaller building blocks makes it easier or less expensive to fabricate that structure. Examples include a wall made of bricks or blocks, the tiling of a floor, the shingles of a roof, and the panes of a window. An example of a decorative brick pattern is shown at left in Figure 5 (group *p*4*g*). A wide variety of patterns have been employed in brickwork [4]. Tiles can be laid in decorative patterns, but individual tiles can also be decorated with a tessellation, as shown at right in Figure 5 (group *cm*). Before plate glass was developed, larger windows were made of smaller pieces of glass supported by lead, as in Figure 6, left (group *p*6*m*). At right in Figure 6, small rhombus-shaped tiles are used to create an optical illusion on a floor in Pompeii (group *p*6*m* if the colors of the tiles are ignored). European churches and cathedrals are a rich source of tessellations used in tiling and stained-glass windows [3].

Figure 5: *Brickwork on a church steeple (left). This particular tessellation suggests an over/under weave. Photographed in the Friesland region of The Netherlands. Tiles photographed in a Mexican imports store in Phoenix, Arizona, USA (right).*

Tessellations also show up in large structures that need strength without the expense and weight of solid construction, such as the struts of a bridge. They can also serve as an decorative elements in architecture, as in the two Islamic tessellations shown in Figure 7 (group *p*6*m* left, and *pmm* right). Islamic design has a uniquely rich and sophisticated tradition of using geometric tessellations [6]. At left in Figure 8 (group *p*4*g)*, a door panel is decorated with a tessellation that is associated with East Asia. This tessellation can be drawn on a grid of squares; i.e., the tile is a polyomino. Another tessellation with a strong cultural association is the Cairo pentagon tessellation, so named because it is used in pavers in Cairo, Egypt [1]. The pentagons in this tessellation are characterized by bilateral symmetry and a pair of right angles. At right in Figure 8 (group *p*4 if the tile colors are ignored), this tessellation is painted across the façade of a hotel in Beijing, China.

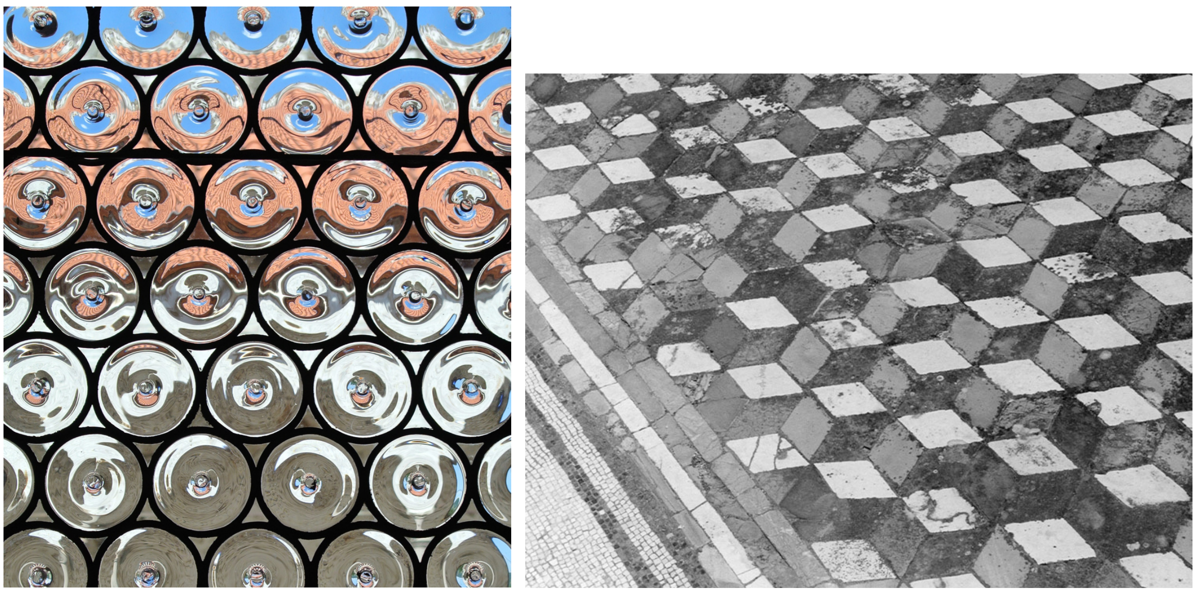

Figure 6: *Leaded glass window (left). Photographed in Visegrád, Hungary. Tiled floor of a Roman villa (right). Photographed in Pompeii, Italy.*

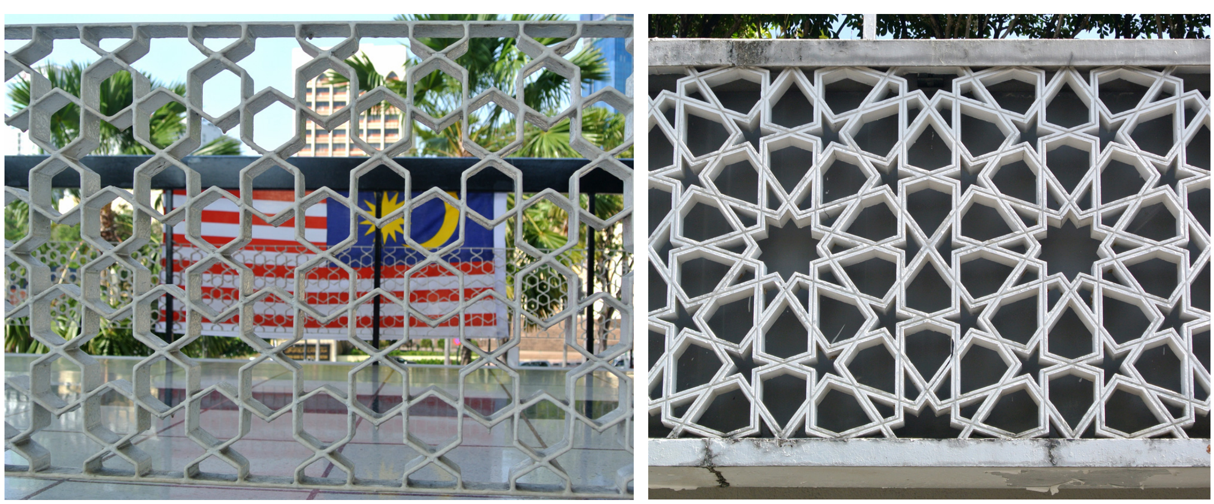

Figure 7: *Two decorative architectural elements that make use of Islamic star tessellations. Photographed at the National Mosque in Kuala Lumpur, Malaysia.*

Figure 8: *A characteristic East Asian tessellation adorns a door panel in the Forbidden City (left). A tessellation commonly associated with Cairo, Egypt makes a bold façade for a hotel in Beijing (right). Photographed in Beijing, China.*

Tessellations have also been used widely in games and puzzles. Common examples include the squares in a chessboard or checkerboard, the divisions of a bingo card, a shuffleboard court, a dart board, and the division of an image into many small pieces in a jigsaw puzzle. There are also numerous puzzles based on polyominoes, polyhexes, etc. The tangram is a set of seven tiles covering a square [7].

Conclusions

A rich diversity of tessellations fill our world in the form of both naturally-occurring and synthetic objects. Learning to recognize and appreciate them expands our understanding of the role of mathematics in shaping the world. In addition to math and art, real-world tessellations integrally relate to human culture, biology, architecture, materials science, and engineering.

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