Folding Pseudo-Stars that are Cyclicly Hinged

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Abstract

Motivated by a (layered) folding dissection of a 2-high {10/3}-star to a 4-high {5/2}-star, we identify an infinite class of folding dissections for pseudo-stars for which the dissections are cyclicly hinged. In addition to folding dissections of 2-high to 4-high pseudo-stars, the class includes folding dissections of 2-high to $(2h)$-high pseudo-stars for any whole number h, as well as folding dissections of $(2h)$-high to $(2h')$-high pseudo-stars where h and h' are whole numbers such that $\gcd(h, h') = 1$. The total number of pieces in each of these folding dissections is the sum of the number of points in both stars.

Introduction

Fifty years ago Harry Lindgren [7] identified, for each natural number k, a trigonometric relationship between a $\{(4k{+}2)/(2k{-}1)\}$-star and a $\{(2k{+}1)/k\}$-star. This led to his lovely family of dissections, with the k-th dissection using $4k{+}2$ pieces. For $k = 2$, Lindgren gave the elegant 10-piece dissection of a {10/3}-star to two {5/2}-stars that we see on the left in Figure 1. In [2, 3], I observed that each of the dissections in that infinite family is swing-hingeable. We see the natural hinging on the right in Figure 1. We can also adapt the dissections to be piano-hingeable, as in [4].

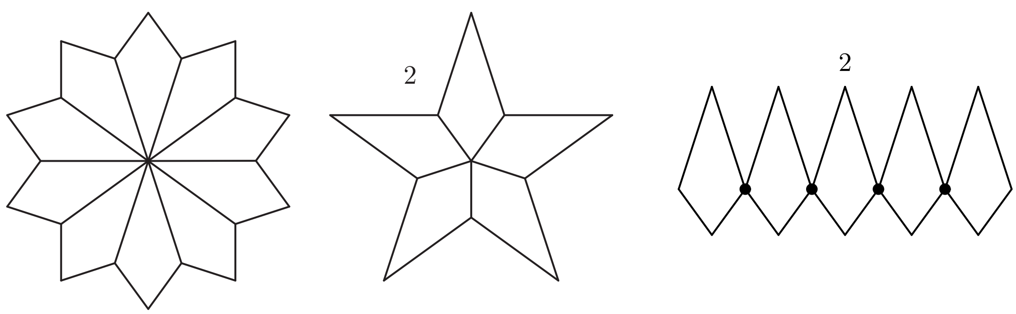

Figure 1 : *A swing-hinged dissection of a {10/3}-star to two {5/2}-stars, with a hinging*

Allowing the $\{(2k+1)/k\}$-star to have twice the thickness of the $\{(4k+2)/(2k-1)\}$-star leads to a folding dissection with just one connected assemblage. In [5, 6], I described a $(6k+3)$-piece folding dissection of a 2-high $\{(4k+2)/(2k-1)\}$-star to a 4-high $\{(2k+1)/k\}$-star. We assume that all pieces are rigid, with $4k+2$ pieces being one level thick, and the remaining $2k+1$ pieces being two levels thick. Figure 2 shows my 15-piece dissection for the case of $k = 2$, where pieces B, E, H, K, and N are the 2-high pieces. The levels of each star are arranged in sequence. A dotted line indicates a fold-hinge that connects two pieces on the same level, while a fold-hinge connecting two pieces on different levels is shown with a row of dots adjacent to the hinge axis. For instance, pieces F and G share a short fold-hinge.

These dissections are so intriguing that I sought to discover other related folding dissections: Are there relatives of these dissections that are cyclicly hinged? (A hinged dissection is *cyclicly hinged* if the pieces

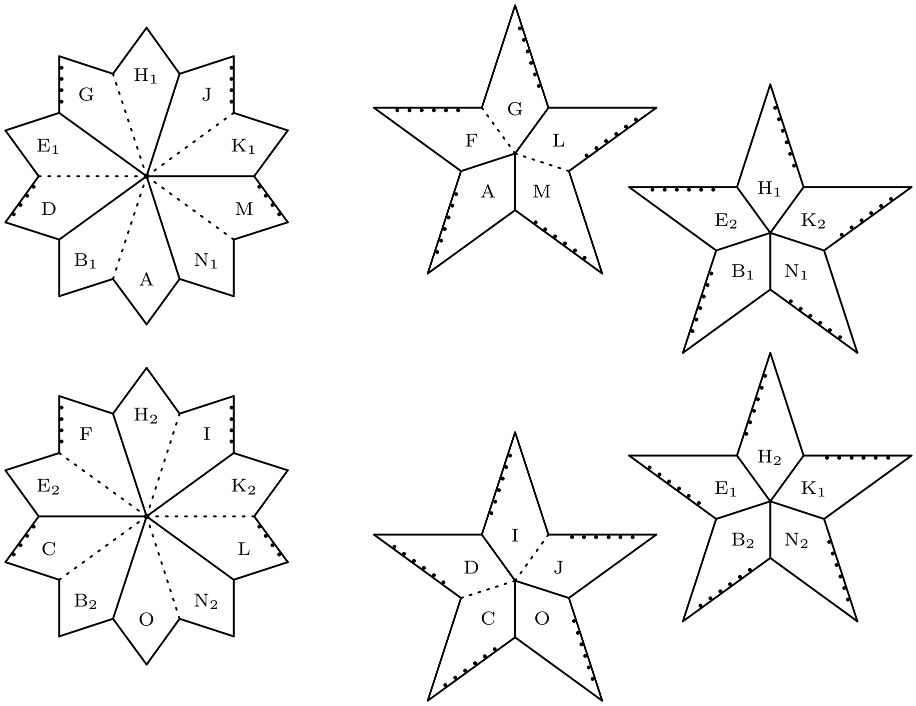

Figure 2 : *Folding dissection of a 2-high {10/3}-star to a 4-high {5/2}-star*

and hinges form a cycle.) Are there related folding dissections of a p-high star to a q-high star where p and $q \neq p$ differ by some factor other than 2?

As we shall see in this article, the answers are yes, if instead of stars we are allowed to use pseudo-stars (as defined on pages 87–89 of [3]). For natural numbers $p \geq 5$ and $1 < q < p/2$, a *star* $\{p/q\}$ has p evenly-spaced points, with each point connected to the q-th preceding point and the q-th following point. A *pseudo-star* $\{p/q\}$ allows q to be any rational number such that $1 \leq q < p/2$. For each point, identify a *pseudopoint* by first finding the $\lfloor q \rfloor$-th point clockwise from our point and then proceeding $q - \lfloor q \rfloor$ further clockwise around a circle that contains all points of the pseudo-star. Similarly identify a pseudopoint counterclockwise from the point. Then draw line segments from each point to its two pseudopoints, delete the portions of those line segments from a pseudopoint to the first line segment that it crosses, and remove portions of line segments that are interior to the resulting figure.

We illustrate this for pseudo-star $\{6/(5/2)\}$ in Figure 3. On the left of the figure, the solid dots are

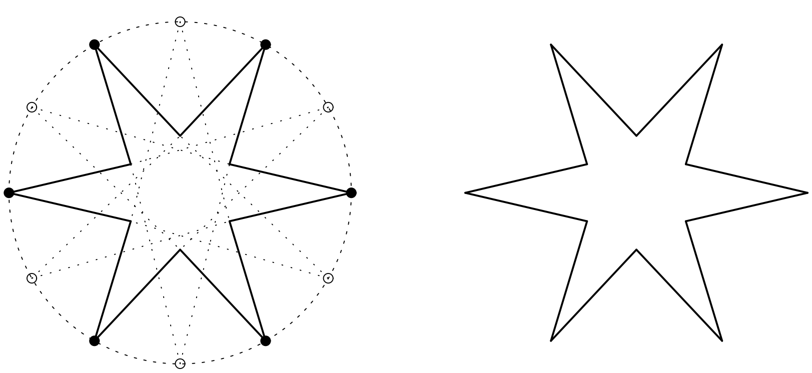

Figure 3 : *Generating the pseudo-star {6/(5/2)}*

points of the pseudo-star, and the open dots are pseudopoints. All points and pseudopoints fall on a circle whose center is at the center of the pseudo-star. Each open dot here represents two pseudopoints that fall exactly between two points. Dotted lines represent portions of line segments that we remove, while solid line segments represent portions of segments that remain. The resulting pseudo-star is on the right. Figure 4 gives a further example, pseudo-star {4/(5/3)}, which has two distinct pseudopoints between every pair of points.

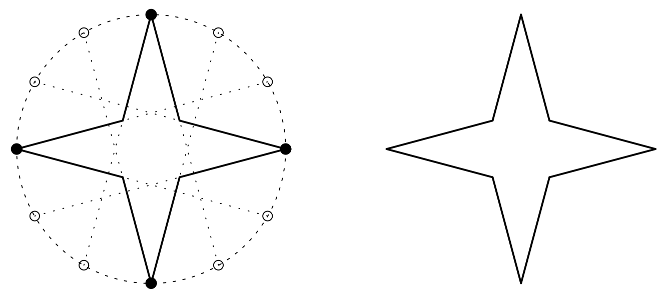

Figure 4 : *Generating the pseudo-star {4/(5/3)}*

By allowing $k > 1$ to be any odd integral multiple of $1/2$, Stuart Elliott [1] extended Lindgren's family of dissections to dissections of a $\{(4k+2)/(2k-1)\}$-star to two $\{(2k+1)/k\}$-pseudo-stars.

Achieving cyclic hingings

Lindgren's original family of star dissections fails to support cyclic hingings because the number of points in a $\{(2k+1)/k\}$-star is odd. The number of fold-hinges on the top side of the $\{(2k+1)/k\}$-star can be at most k, since otherwise three consecutive pieces on the top side of the $\{(2k+1)/k\}$-star would be connected by fold-hinges, and that would cause there to be three levels in the $\{(4k+2)/(2k-1)\}$-star. However if $k > 1$ is an odd integral multiple of $1/2$, then the corresponding 4-high pseudo-star $\{(2k+1)/k\}$ will have an even number of points and thus not cause a problem.

Substituting $k = 5/2$ into $\{(4k+2)/(2k-1)\}$ and $\{(2k+1)/k\}$ from Lindgren's extended family, we get a 2-high star $\{12/4\}$ and a 4-high pseudo-star $\{6/(5/2)\}$. Figure 5 displays these figures with an appropriate hinging. Starting with piece A on the top level of the 2-high star, the pieces are hinged in clockwise order around that star, with the last piece (piece R) ending up beneath piece A, so that we can close the cycle with one more hinge. With piece A also on the top level of the 4-high star, the sequence of pieces proceeds down through the levels, shifts over one position clockwise on the bottom level, and then proceeds up again, and repeats until the last piece lands next to piece A on the top level.

To confirm that the star actually does fold into the pseudo-star without obstruction, I built a model consisting of laser-cut pieces of cherry wood, hinged with a clear tape. The model looks great and executes the desired motion perfectly! From the 12-pointed star, we can fold the cycle of pieces "inside out" to produce the 6-pointed pseudo-star. We require the height of a level to be sufficiently small, so that it does not interfere with the folding. A height of a level that is at most $1/80$ of the pseudo-star's diameter is certainly satisfactory.

We carefully describe a folding from the 4-high pseudo-star $\{6/(5/2)\}$ to the 2-high star $\{12/4\}$: Simultaneously fold piece F against piece G, piece L against M, and piece R against A, achieving a cycle of three stacks: pieces D through I, pieces J through O, and pieces P through R to A through C. This leaves a triangular-shaped hole between the stacks, with three short hinges left open. Then fold out simultaneously along the hinges between pieces G and H, between J and K, between B and A, and between P and Q, achieving a cycle of four stacks: pieces B through G, pieces H through J, pieces K through P, and pieces

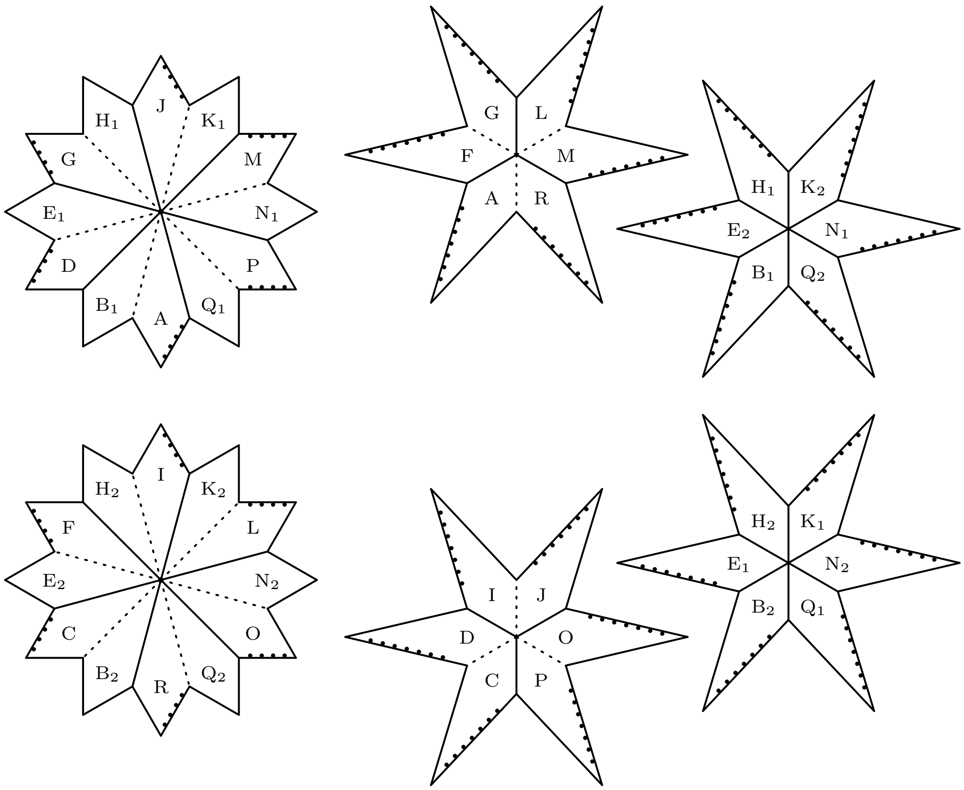

Figure 5 : *A cyclic hinging of a 2-high* $\{12/4\}$ *to a 4-high* $\{6/(5/2)\}$

Q through R to A. This leaves a rectangular-shaped hole between the stacks, with four long hinges left open. We then mash the rectangular hole down, achieving two side-by-side stacks, pieces B through J, and pieces K through R to A. Finally we simultaneously fold the two stacks out to get the 12-pointed star.

It is then easy to see that the cyclic hinging works for each $k > 1$ that is an odd integral multiple of $1/2$. Just as in my original family of folding dissections, and also apparent from the figure, the second and third levels of the 4-high pseudo-star $\{(2k{+}1)/k\}$ consist of 2-high pieces, with these pieces forming every second point in the 2-high star $\{(4k{+}2)/(2k{-}1)\}$. The dissection has $6k{+}3$ pieces, and the hinging both is cyclic and has $(k{+}1/2)$-fold rotational symmetry.

When one height is not double the other

My approach also works when the $(2k{+}1)$-pointed star is $(2h)$-high for any natural number $h > 1$, as long as $k > 1$ is an odd integral multiple of $1/2$. In this case the dissection is of a 2-high $\{(2hk + h)/(hk - h/2)\}$-pseudo-star to a $(2h)$-high $\{(2k{+}1)/(k{+}1/2{-}1/h)\}$-pseudo-star.

The number of points times the height of one pseudo-star must equal the number of points times the height of the other pseudo-star. Also, for a given pseudo-star, the product of the angle of a point times the number of points times the height must equal the product of the angle of a point times the number of points times the height of the other pseudo-star. For pseudo-star $\{p/q\}$, each point has angle of $(180 - 360q/p)$ degrees.

Suppose that the height of one pseudo-star is 3 times the height of the other, so that one pseudo-star has $6k{+}3$ points and the other pseudo-star has $2k{+}1$ points. We then substitute 3 for h, yielding the family of dissections that converts a 2-high $\{(6k{+}3)/(3k{-}3/2)\}$-pseudo-star to a 6-high $\{(2k{+}1)/(k{+}1/6)\}$-pseudo-

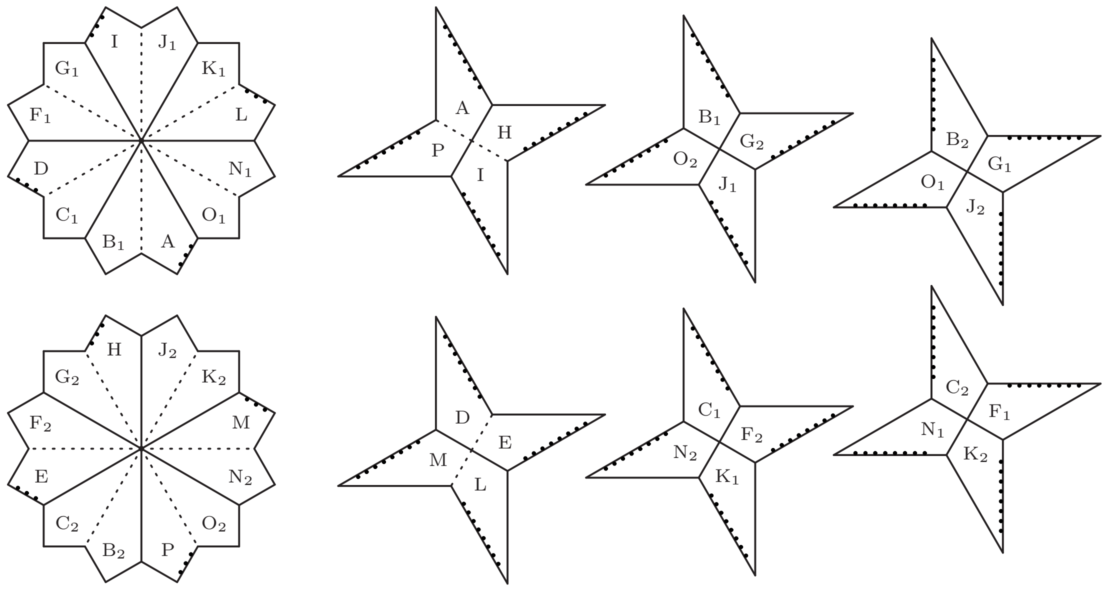

Figure 6 : *A cyclic hinging for folding a 2-high* $\{12/3\}$ *to a 6-high* $\{4/(5/3)\}$

star, for each $k > 1$ that is an odd integral multiple of $1/2$. With $k = 3/2$ we have a 2-high 12-pointed star and a 6-high 4-pointed pseudo-star, with the points of the first equal to $90°$, and the points of the second equal to $30°$. Then we will have $q = 3$ for the 12-pointed star and $q = 5/3$ for the 4-pointed pseudo-star. Figure 6 shows the 16-piece cyclicly folding dissection and Figure 7 shows photos of my cherry wood model for the corresponding stars.

Figure 7 : *Cherry wood model for a 2-high* $\{12/3\}$ *to a 6-high* $\{4/(5/3)\}$*)*

Suppose that the height of one pseudo-star is 4 times the height of the other pseudo-star. We then apply the appropriate substitution, yielding the family of dissections that converts a 2-high $\{(8k+4)/(4k-2)\}$-pseudo-star to an 8-high $\{(2k+1)/(k+1/4)\}$-pseudo-star, for each $k > 1$ that is an odd integral multiple of $1/2$. For example, if $k = 3/2$ we have a 2-high 16-pointed pseudo-star and an 8-high 4-pointed pseudo-star, the points of the first equal $90°$, and the points of the second equal $22.5°$. Then for the 16-pointed star, we have $q = 4$ and for the 4-pointed star, $q = 7/4$. In Figure 8 we show the 20-piece cyclic folding dissection.

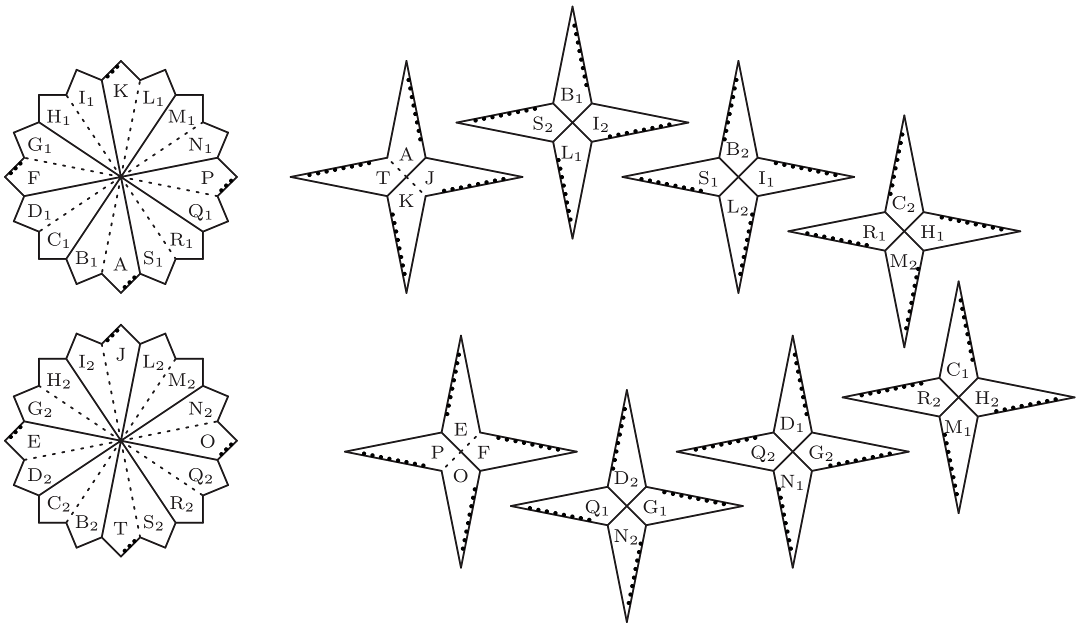

Figure 8 : *A cyclic hinging for folding a 2-high* $\{16/4\}$ *to an 8-high* $\{4/(7/4)\}$

Again, a cyclic hinging works for each $k > 1$ that is an odd integral multiple of $1/2$. The second level through the $(2h{-}1)$-th level of the $(2h)$-high pseudo-star $\{(2k{+}1)/(k{+}1/4)\}$ are filled in with 2-high pieces, with these pieces forming all but every h-th point in the 2-high star $\{(2hk{+}h)/(hk{-}h/2)\}$. The dissection has $(h{+}1)(2k{+}1)$ pieces. Starting with piece A on the top level of the 2-high star, the pieces are hinged in clockwise order around that star, with the last piece ending up beneath piece A and hinged to that piece. With piece A also on the top level of the 8-high pseudo-star, the sequence of pieces goes down through the levels, shifts over one position counterclockwise on the bottom level, and then up again, repeating until the last piece lands next to piece A on the top level and is hinged to that piece. The hinging is cyclic and has $(k{+}1/2)$-fold rotational symmetry.

When neither pseudo-star is of height 2

What happens when neither pseudo-star is of height 2? Let one pseudo-star $\{p/q\}$ be $(2h)$-high and let the other $\{p'/q'\}$ be $(2h')$-high, where $\gcd(h, h') = 1$. Choose $p = h'(2k{+}1)$ and $p' = h(2k{+}1)$. Then choose $q = p/2 - p/p'$ and $q' = p'/2 - p'/p$. For each $k > 1$ that is an odd integral multiple of $1/2$, p and p' are even. This allows us to start at a piece on the top level of one pseudo-star, go down to the bottom of that point, shift to the next point, come up to the top, and then repeat that movement, producing two points at a time. With an even number of points, we can close the cycle. Simultaneously, we can move through the other pseudo-star, from top to bottom and back up with the next point, and repeat. Since the number of points in that figure is also even, we close the cycle in that figure too.

In these traversals we must shift from one piece to the next piece whenever we reach the top or the bottom of a pseudo-star. Thus the number of shifts is $p + p'$, the total number of points in both pseudo-stars. Consider a dissection of a 4-high pseudo-star to a 6-high pseudo-star, when $k = 3/2$. This results in $h = 2$, $p = 12$, $q = 9/2$, $h' = 3$, $p' = 8$, and $q' = 10/3$, thus producing the dissection of a 4-high $\{12/(9/2)\}$ to a 6-high $\{8/(10/3)\}$ in Figure 9. There are twenty pieces, of which eight (A, C, F, H, K, M, P, R) are 1-high, eight (D, E, I, J, N, O, S, T) are 3-high, and four (B, G, L, Q) are 4-high. This folding dissection is cyclicly hinged and works wonderfully, as I have demonstrated with an actual 20-piece wooden model.

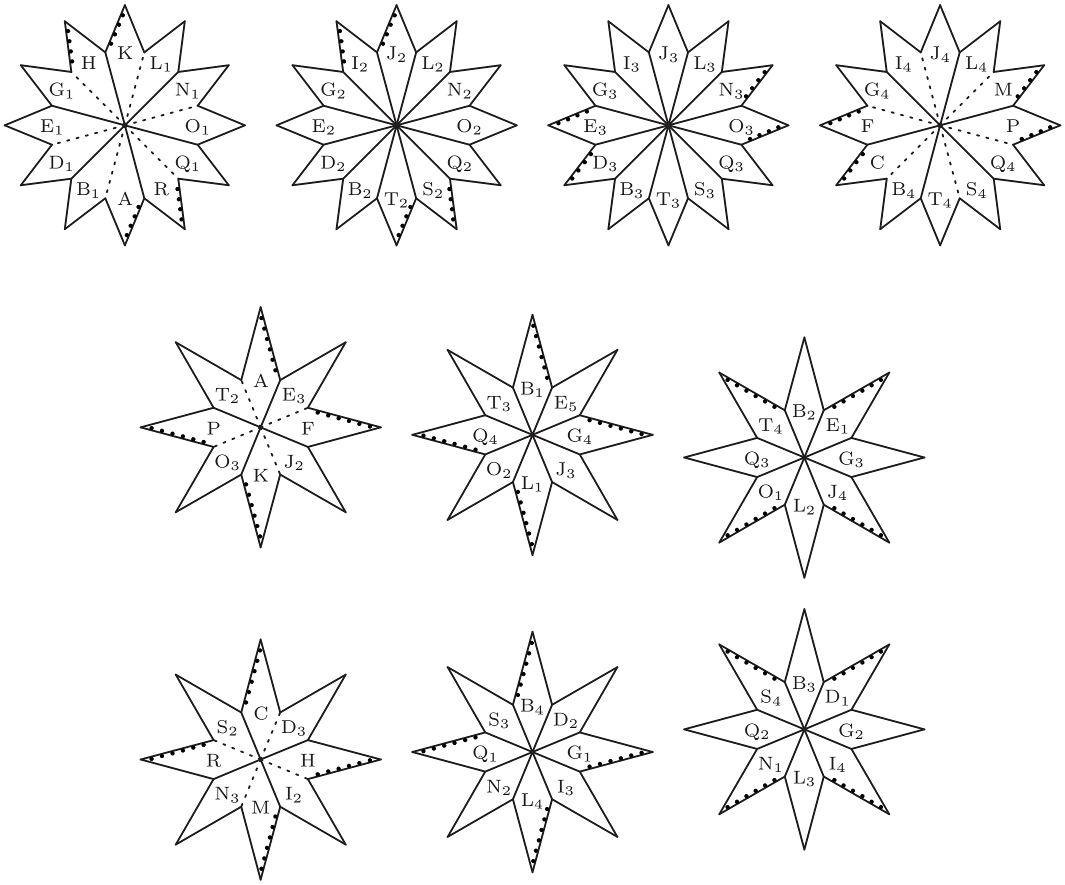

Figure 9 : *Cyclicly folding a 4-high* $\{12/(9/2)\}$ *to a 6-high* $\{8/(10/3)\}$

Figure 10 shows photos of my cherry wood model for the corresponding stars.

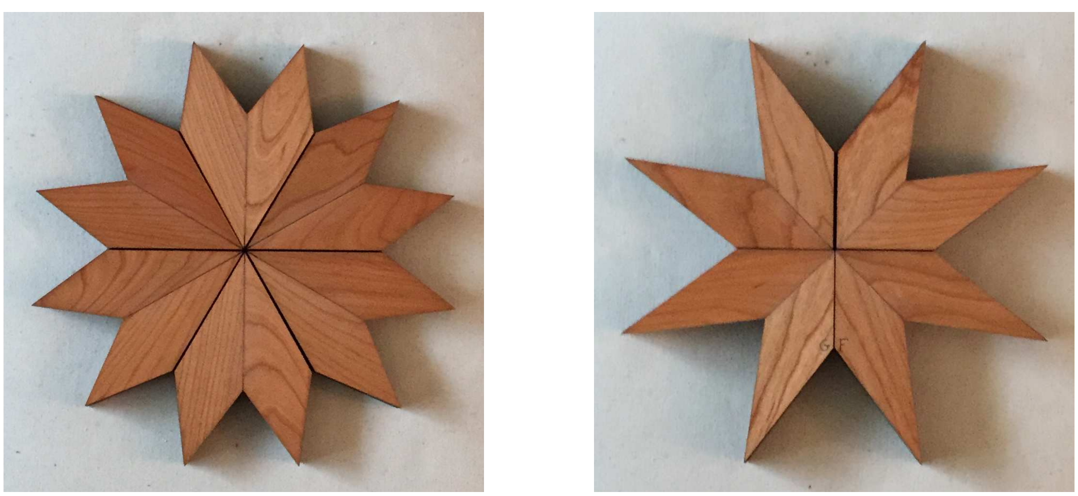

Figure 10 : *Cherry wood model for a 4-high* $\{12/(9/2)\}$ *to a 6-high* $\{8/(10/3)\}$

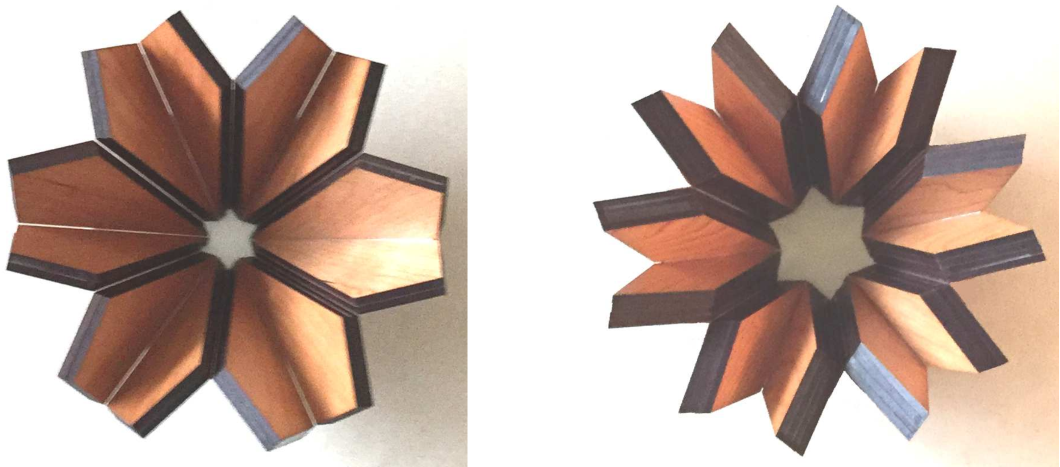

Figure 11 : *Partially folded-out models of Figures 6 and 7, and of Figures 9 and 10*

Conclusion

We have progressed from folding dissections for an infinite family of stars to a richer family of pseudo-stars. While we previously allowed one parameter k to range over an infinite sequence, we now use parameters k, h, and h'. Then we can identify cyclic hingings that convert one pseudo-star to another by folding the cyclicly-hinged assemblage "inside out." Enjoy Figure 11, which shows partial inside-out folding of the models of Figures 6 and 7, and of Figures 9 and 10. There are also acyclic folding dissections of a broader class of pseudo-stars, including pseudo-stars with an odd number of points or with $k = 1/2$. Since the folding dissections in those cases are not as elegant as those already shown, I have not included them here.

Acknowledgement

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