Some Hyperbolic Fractal Tilings

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Abstract

The concepts of fractal tiling and hyperbolic tiling are combined to create novel fractal surfaces in Euclidean three-space. Paper folding and computer modeling are used to create these constructs. We show examples using triangular, trapezoidal, and dart-shaped prototiles. Smaller tiles deflect out of the plane of adjacent larger tiles, resulting in non-planar surfaces, the shapes of which are dependent on the rules governing the sign of the deflections as well as their magnitudes. These constructs are an intriguing blend of organic and geometric character and in some cases bear marked resemblance to natural leaves.

Fractal Tilings and Hyperbolic Tilings

We have previously described a variety of fractal tilings, in which tiles are adjacent to larger and smaller tiles that are similar [2-3, 6]. Typically, the starting point is a small group of first-generation tiles of a single type, about which smaller tiles are arranging in edge-to-edge fashion according to some matching rule. These fractal tilings contain singular points and are of finite extent in the Euclidean plane, with boundaries that are fractal curves.

Hyperbolic tilings are commonly depicted using the Poincaré disk model of the hyperbolic plane [1], and they satisfy the condition that the sum of the angles meeting at each vertex exceeds 360°. They can also be constructed in Euclidean three-space, in which case a rumpled and chaotic surface results [4]. Hyperbolic surfaces of limited extent have been created using crocheting and knitting, often with esthetically-pleasing results [9].

We report here on fractal tilings in which the sum of the angles meeting at a vertex is greater than 360 degrees; i.e., hyperbolic fractal tilings. These have been constructed in Euclidian three-space using both computer modeling and paper models. These structures are two-dimensional surfaces of finite extent living in three-dimensional space and do not conform to any rigorous definition of the hyperbolic plane. They are constructed of flat two-dimensional tiles, but they don't tile any particular surface. The goal of this work is to determine what sort of novel and esthetically-pleasing fractal surfaces can be constructed using this technique. Of particular interest are geometric structures that mimic biological structures. All of the images, with the exception of Figures 1 and 7, were generated in Mathematica.

Folded Two-dimensional Fractal Tilings

Perhaps the simplest and most insightful starting point for exploring this topic is to experiment with folding some two-dimensional fractal tilings. A simple fractal tiling is shown in Figure 1, for which each tile is an isosceles right triangle [6]. A tiling in which every tile is similar is referred to as a single-prototile tiling, and all of the constructs described here are of this type. The long edges of two second-generation tiles are matched to the short edges of the first-generation tile along the entire length of each edge. A tiling that obeys this restriction is called "edge-to-edge", and this forces the scaling between successive generations to be the square root of two for this fractal tiling. A surface residing in the three-dimensional space can be created from this two-dimensional structure simply by deflecting tiles along the shared edges.

Creation of a hyperbolic fractal tiling from a flat piece of paper is problematical due to the fact that the angle sum at each vertex should exceed 360°. In Figure 1, this has been overcome for the first few vertices by laminating four pieces of paper together. The angle sum at the apex of the first triangle is then 540°, while the sum at the apexes of the two second-generation triangles is 450°. Note that the construction has been carried out to six generations, at which level no additional vertices have angle sums exceeding 360°.

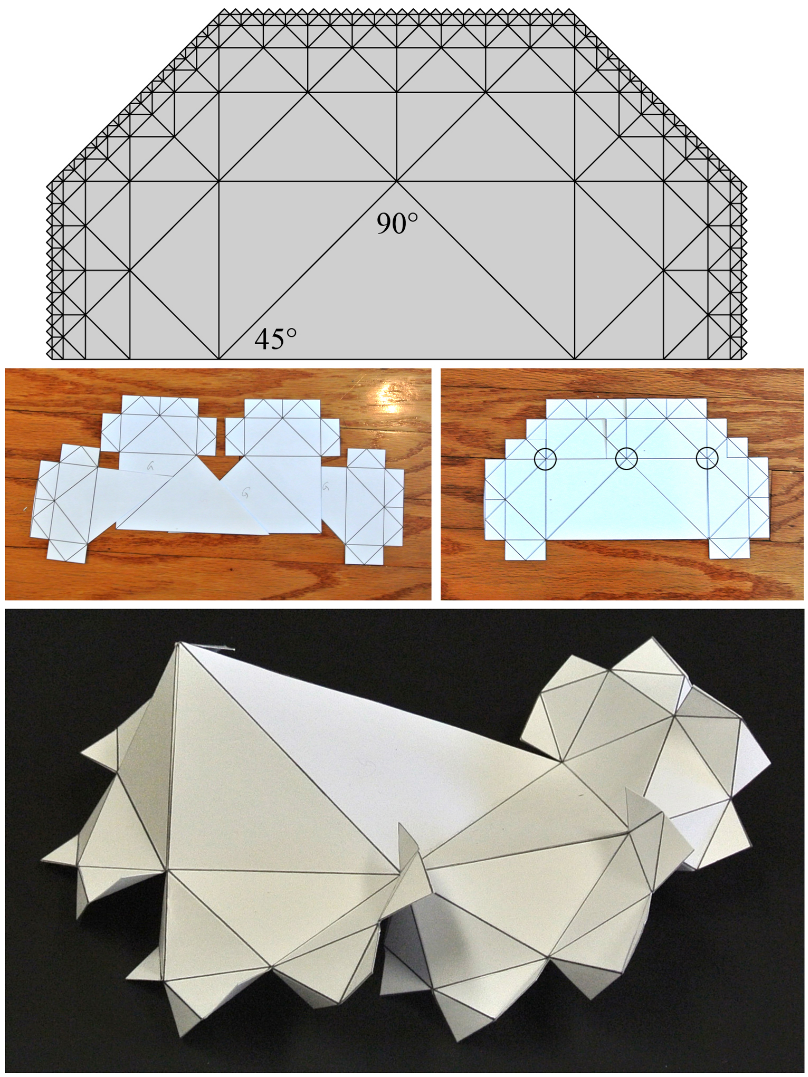

Figure 1: *A hyperbolic fractal tiling created by folding a planar fractal tiling. The starting fractal tiling is shown at top. At middle left is a photograph showing four panels to be glued together. After gluing, a flat fractal tiling is obtained in which three of the vertices have angle sums exceeding 360° (the circled vertices in the photograph at middle right, at which triangular tiles overlap). A hyperbolic fractal tiling is obtained by folding along the edges of the structure, as shown at bottom.*

If this "hyperbolic paper" is then folded according to a simple rule by which the left triangle coming off any large triangle deflects up and the right down, the structure shown at bottom in Figure 1 is obtained. Note that left and right triangles still deflect according to the rule if the entire structure is flipped over. The dominant features of this structure are a pair of what could be described as spiraling pyramids without bases coming off the first-generation triangle. Smaller spiraling features occur repeatedly throughout the structure.

If carried through an infinite number of iterations, the boundary of this structure should be some sort of fractal space curve. It's not practical to construct a large number of generations by folding paper, and the limited construction of Figure 1 only hints at what that boundary curve might look like. The construction was carried further in Mathematica, yielding the construct of Figure 2 after ten generations. Due to the slow shrinkage rate of the tiles, the smallest triangles are still relatively large in this version.

In the two-dimensional fractal tilings discussed previously, prototiles and angles were carefully chosen such that the tiles always come together in edge-to-edge fashion when smaller tiles wrap around to meet each other. In the case of a three-dimensional fractal tiling, this is much more difficult to achieve, as the tiles are now taking on different orientations in space rather than being restricted to the plane. As a result, interpenetrations of smaller tiles and large tiles generally occur at some point in these structures. The arrow in Figure 2 points out one of these areas. No examples have been observed to date of single-prototile edge-to-edge hyperbolic fractal tilings that do not intersect at some point.

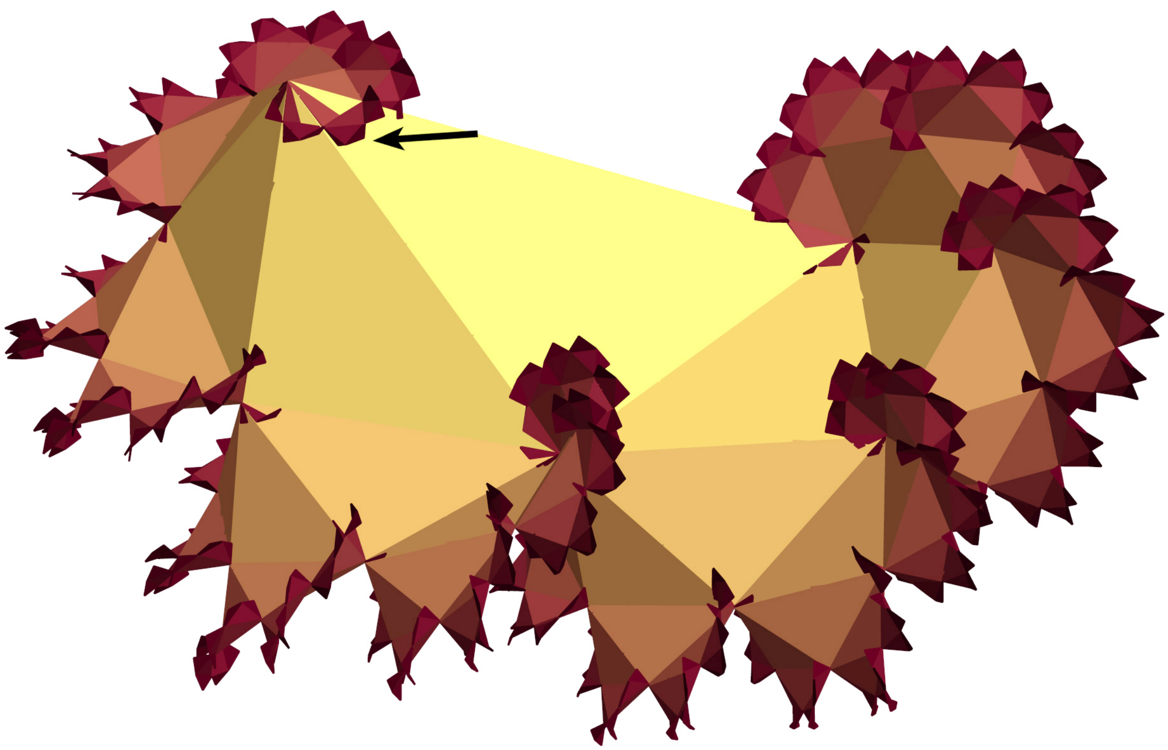

Figure 2: *A hyperbolic fractal tiling constructed from isosceles right triangle prototiles. The construction has been carried through ten generations of tiles.*

Fractal tilings with similar matching rules, but constructed from trapezoids [6], have also been examined. In the example of Figure 3, the angles of the prototile are all 60° or 120°. Three smaller tiles mate to a larger tile, with a single long edge matching each of the short edges. The scaling factor is 0.5. The planar fractal tiling is shown at upper left in Figure 3, where all of the trapezoids are shown through six generations, including the extra ones at the inner vertices. To bring the structure out of the plane, the middle trapezoid coming off each larger trapezoid is deflected up and the outer two deflected down. Note that this is reversed if the structure is flipped over, so the structure looks different from top and bottom, in contrast to the structure in Figure 1. Different values can be employed for the magnitude of the deflection, of course, and deflection angles of 10°, 30°, and 50° are shown in Figure 3. The 30° structure taken through nine generations is shown in Figure 4. Due to the more rapid shrinkage rate between successive generations, the smallest tiles are smaller in this case than for the isosceles triangle case discussed above. The structure in Figure 4 is somewhat reminiscent of the leaves of some plants.

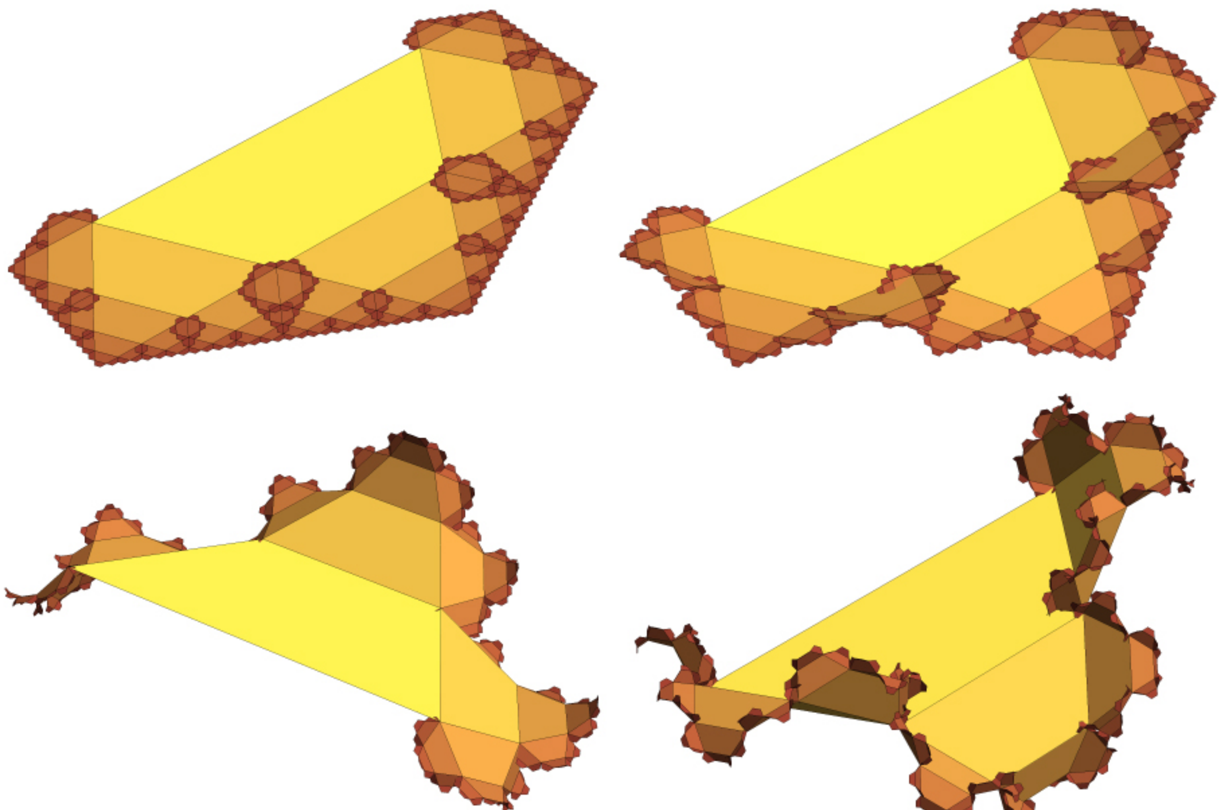

Figure 3: *Hyperbolic fractal tilings constructed from a trapezoidal prototile, taken through six generations. At top left, the starting two-dimensional fractal tiling is shown. At upper right, the structure is shown that results from deflecting each smaller middle tile downward by 10° and each pair of smaller left and right tiles up by 10°. At lower left and right, similar structures are shown, but using deflection angles of 30° and 50°, respectively. For 30°, the underside is shown.*

Dart-shaped Prototiles

Kite- and dart-shaped prototiles work well for two-dimensional fractal tilings [2]. Three hyperbolic fractal tilings generated form dart-shaped prototiles are shown in Figure 5. To make a structure that resides in three dimensional space, the interior angle between the two long edges of the prototile is chosen to be larger than half the exterior angle between the two short edges. This forces a pair of smaller tiles set into that exterior angle to buckle either upward or downward. For the structures shown in Figure 5, two pairs of darts that are descended from the same grandparent tile (two generations larger) buckle up (left pair) and down (right pair). As was the case for the structure of Figure 1, the rule is the same from top and bottom. Note that the scaling factor, angles of the dart, and dihedral angles of the hyperbolic fractal tiling are all interdependent, so that selecting one of them determines the other two.

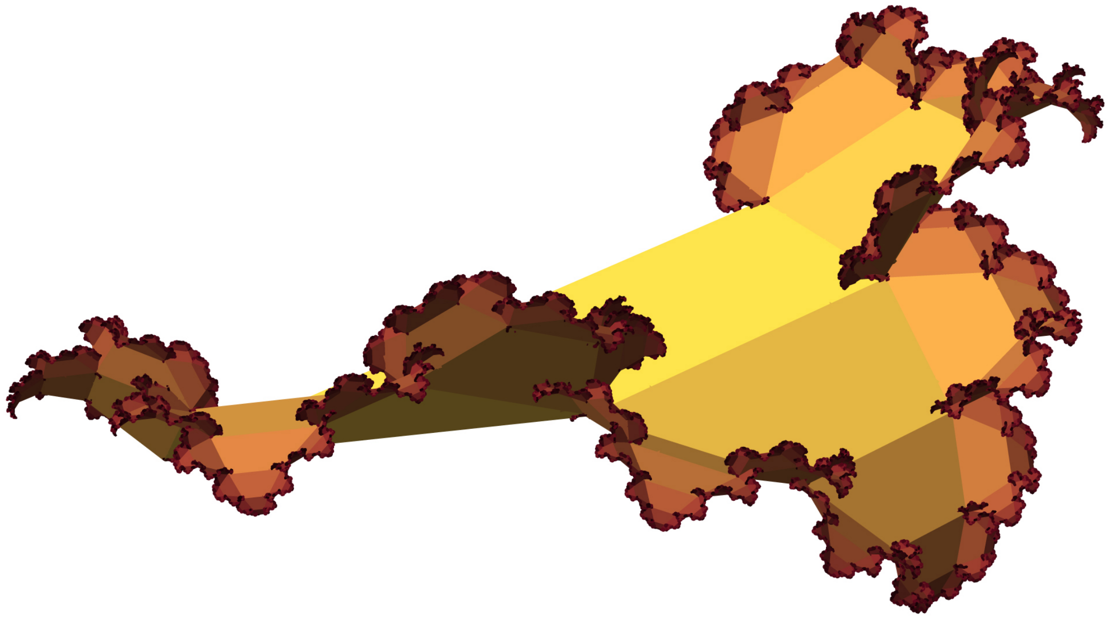

Figure 4: *The same structure shown in Figure 3, with a deflection angle of 30° and carried through nine generations.*

Three different scaling factors are contrasted in Figure 5, in which the structures are carried through six generations. The generation at which interpenetration of tiles first occurs depends on the scaling factor. It happens earliest for large scaling factors, and is observed in the sixth generation for the structure shown at bottom in Figure 5.

The fractal dimension of the boundary curves for the structures in Figure 5 can be computed using the self-similarity notion of fractal dimension as $D = (\log a) / (\log 1/s)$, where a is the number of smaller copies the boundary is broken into when scaled by s [7]. For the structures of Figure 5, $a = 2$ and $s \approx 0.57$, 0.62, and 0.71, resulting in $D \approx 1.23$, 1.45, and 2.02 respectively. As expected, the more "crinkly" the surface, the larger the fractal dimension.

The middle structure of Figure 5 is shown through nine generations in Figure 6 from three different angles. The boundary of the structure has a chaotic and organic appearance. The organic character of the portion of the surface defined by the smaller generations contrasts nicely with the geometric angularity of the surface defined by the larger generations. The structure of Figure 6 bears a marked resemblance to green-leaf lettuce, a natural hyperbolic surface (Figure 7). Other leafy plants such as kale exhibit similar hyperbolic surfaces [8].

Another set of matching rules was employed with the same prototile to generate the structure shown in Figure 8. In this case, all of the tiles in a given generation deflect in the same direction, but that direction alternates with successive generations. This leads to a fractal hyperbolic tiling with mirror symmetry about a plane dividing the structure down the middle.

Conclusions

We have shown examples of hyperbolic fractal tilings constructed using triangular, trapezoidal, and dart-shaped prototiles. Smaller tiles deflect out of the plane of adjacent larger tiles, resulting in surfaces that reside in three-dimensional space, the shapes of which are dependent on the rules governing the signs of the deflections as well as their magnitudes. These constructs are an intriguing blend of organic and geometric character and in some cases bear marked resemblance to natural leaves.

Future work could involve exploring other prototiles and matching rules. One goal would be to find single-prototile examples that never interpenetrate. Developing fractal curves provide one possible avenue to this end [5], though our initial explorations have resulted in structures with more than one prototile. The fractal space curves described by these structures could also be examined in more detail.

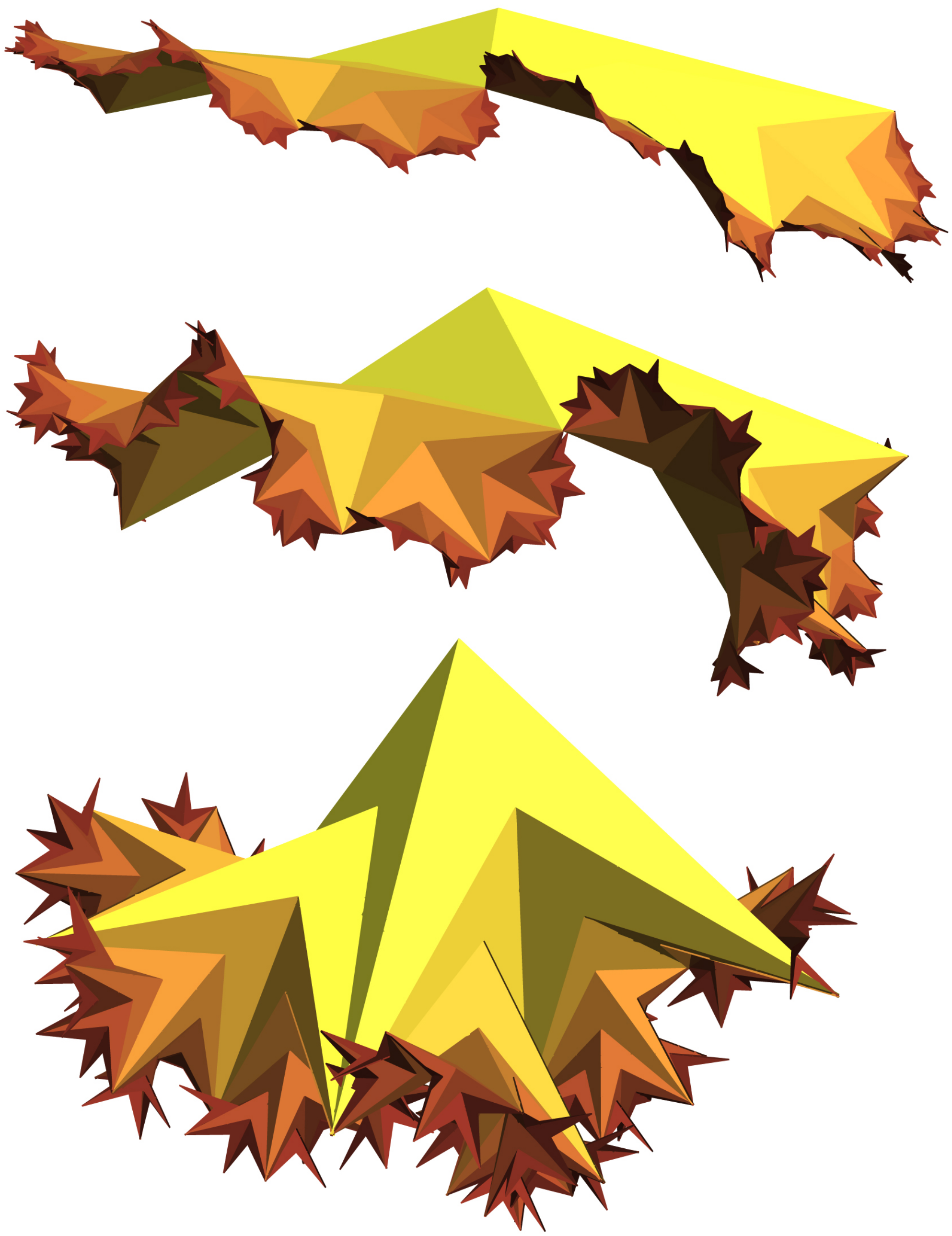

Figure 5: *Hyperbolic fractal tilings constructed from dart-shaped prototiles through six generations. The scaling factors between successive generations are, from top to bottom, ≈ 0.57, ≈ 0.62, and ≈ 0.71.*

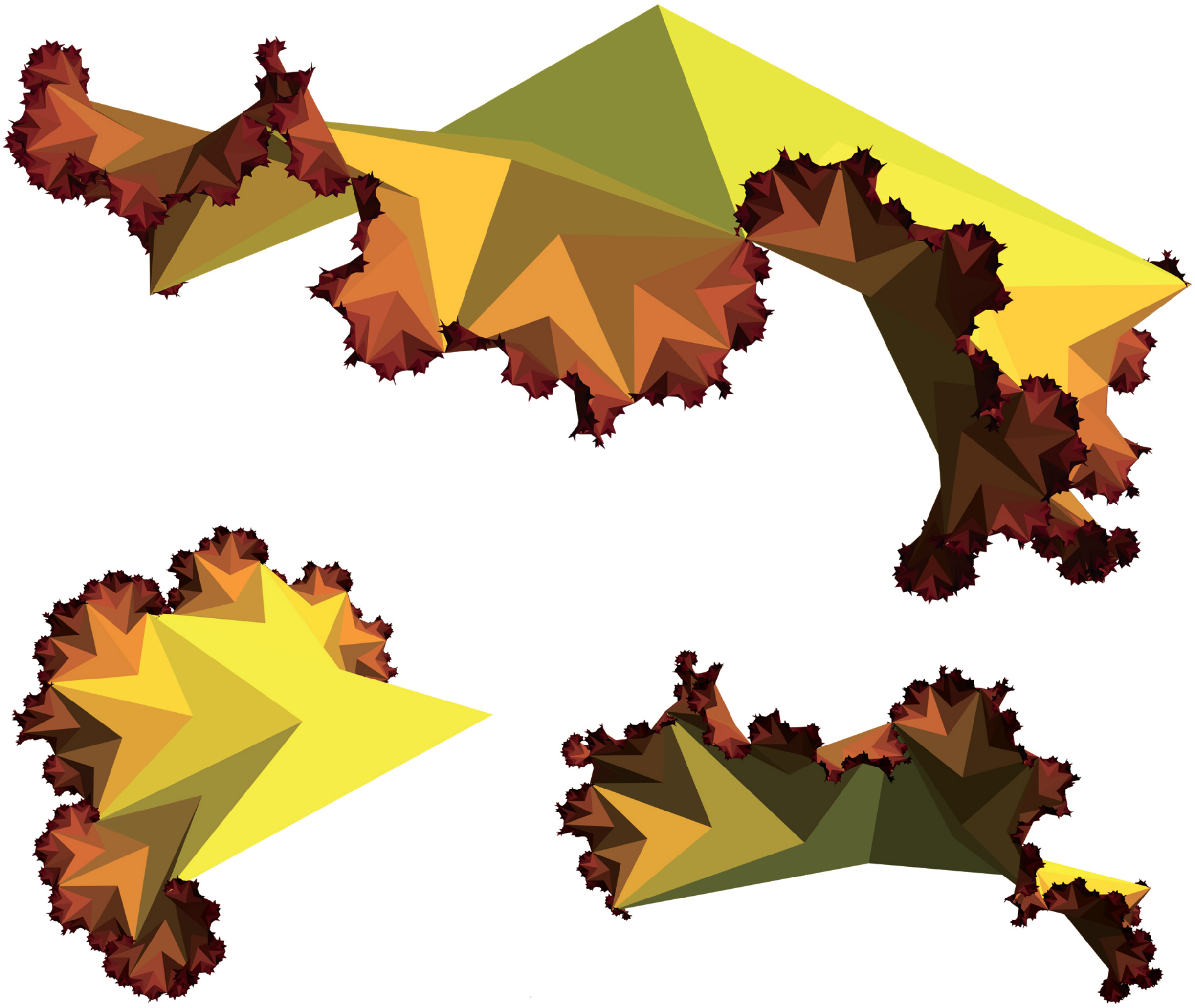

Figure 6: *A hyperbolic fractal tiling constructed through nine generations from dart-shaped prototiles, shown from three different angles.*

Figure 7: *Green leaf lettuce, a hyperbolic surface found in nature.*

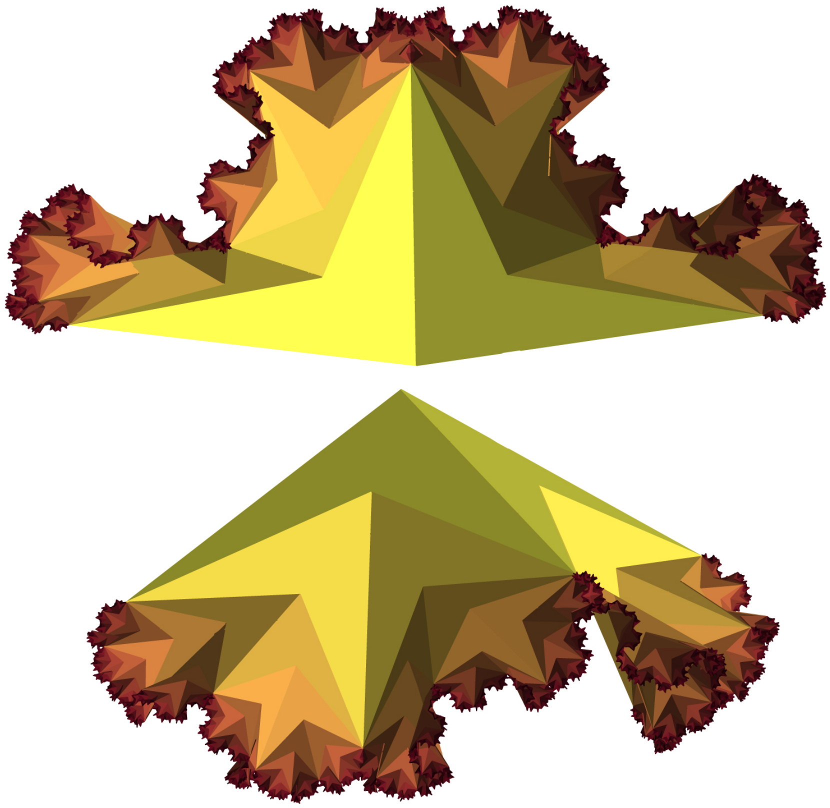

Figure 8: *Two views of a hyperbolic fractal tiling constructed from a dart-shaped prototile using a different matching rule compared to Figure 4. Both pairs of tiles deflect in the same direction, but that direction alternates between successive generations.*

References

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