Seeing a Fundamental Theorem

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Abstract

The *Geometric Modulus Principle*, developed by Bahman Kalantari, provides a powerful geometric means for understanding the Fundamental Theorem of Algebra. Given a complex polynomial and any point in the complex plane, the principle gives a complete characterization of directions of ascent and descent for the modulus of the polynomial at that point. This characterization has a simple and compelling visual representation. By making a contour plot of the modulus of the polynomial, and including only those level curves that pass through the polynomial's critical points, the Geometric Modulus Principle comes to life. The Fundamental Theorem of Algebra is a straightforward consequence, and the visualizations stand on their own as portraits of the polynomials in question.

The Geometric Modulus Principle

The Fundamental Theorem of Algebra states that every nonconstant complex polynomial has a root. While there are dozens of proofs, it is possible to summarize a main line of reasoning common to many of them. One needs to first understand the concept of the *modulus* of a complex number $z = x + iy$. Denoted $|z|$ and sometimes called the absolute value, it is just the distance from z to the origin, or $\sqrt{x^2 + y^2}$. With that tidbit in hand, here is how many proofs of the Fundamental Theorem go: Given a complex polynomial $p(z)$, one first shows that its modulus $|p(z)|$ (a nonnegative real number for each input z) must attain a minimum value on the interior of some disk in the complex plane. It is then argued (in different ways, depending on the proof) that if the complex number z_0 where the minimum is attained is not a root, then from this point there must exist a *direction of descent*, a ray emanating from z_0 along which the modulus of the polynomial must get lower still. Since that is impossible, z_0 must in fact be a root.

The Geometric Modulus Principle (GMP) was developed by Bahman Kalantari and reported in 2011 [1]. It completely characterizes the directions of ascent and descent from *any* given point z_0 for the modulus of a complex polynomial $p(z)$, whether that point is a root, a critical point, or anything else. Essentially, if z_0 is a root, then every direction from z_0 is a direction of ascent for the modulus of p. If z_0 is not a root, let k be the smallest positive integer where the kth derivative $p^{(k)}(z_0)$ is nonzero. (Such a k must exist since $p^{(n)}(z)$ will be a nonzero constant when n is the degree of p.) The GMP says that a small disk centered at z_0 will be equally divided into $2k$ alternating sectors of ascent and descent. That is, if a ray in the complex plane is drawn from the point z_0, and if the ray lies in an ascent sector, the modulus of $p(z)$ will increase, at least initially, as one moves along the ray away from z_0. Similarly, the modulus will decrease if one moves along a ray emanating from z_0 that lies in a descent sector. In particular, the principle guarantees that directions of descent must exist for nonroots z_0. In other words, the Fundamental Theorem of Algebra is a direct consequence. An elementary exposition of the GMP is provided in [2].

Beyond its utility, the principle has intrinsic geometric beauty. It provides a fundamental characterization of how the modulus of a complex polynomial behaves. Imagine a contour plot of the modulus of a complex polynomial. That is, at each point z in the complex plane, we plot the value of the nonnegative real number $|p(z)|$. We can render these values like altitudes in a topographical map, so that contour lines (or level curves) represent sets of points z where $|p(z)|$ is constant—neither ascending nor descending. An example is in order. Consider $p(z) = z^3 - 1$. A 3D plot of the modulus (with the value of the modulus plotted on the vertical axis) is shown in Figure 1, and the corresponding contour plot is shown in Figure 2. Pay particular attention to the point in the center of the contour plot. That point is the origin.

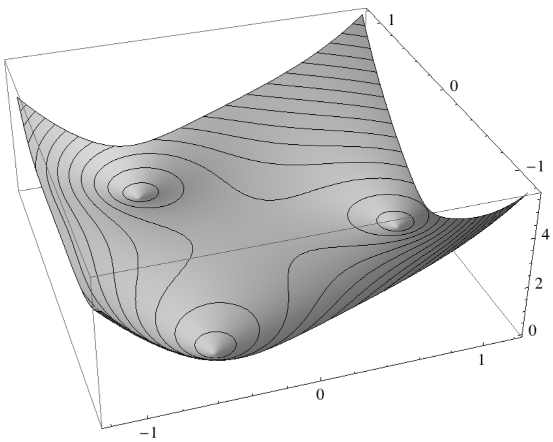

Figure 1: *3D Modulus plot of* $p(z) = z^3 - 1$.

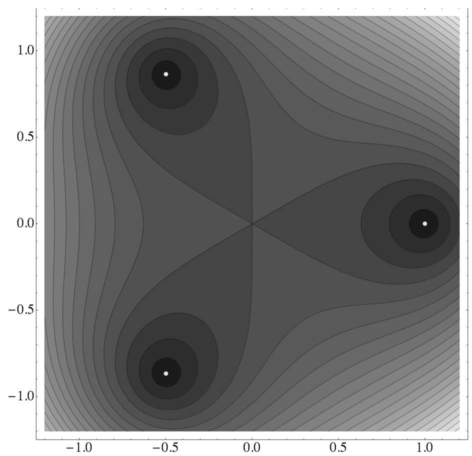

Figure 2: *Modulus contour plot of* $p(z) = z^3 - 1$.

We see that $p(z)$ has three roots (the only points where the modulus is zero), and a critical point at the origin—the solution to $p'(z) = 3z^2 = 0$. So let's see what the GMP says about the critical point $z_0 = 0$. The smallest positive integer k for which $p^{(k)}(0)$ is nonzero is $k = 3$, since $p''(z) = 6z$, and $p'''(z) = 6$. So the GMP states that a disk centered at 0 in the complex plane will be divided equally into $2k = 6$ alternating ascent and descent sectors. That's what the contour lines show in the center of the plot in Figure 2; the level curve where the modulus assumes the value 1 passes through the origin, and it perfectly demarks the boundary lines between the six alternating ascent and descent sectors. The (darker) descent sectors lead downward toward the three roots.

In general, given a polynomial, consider a critical point that is not a root. In the language used to describe the GMP, this is a point with $k > 1$. Now imagine the level curve that passes through this critical point. It traces out the boundary between the ascent and descent sectors of a disk centered at the point. The Geometric Modulus Principle says that if we zoom in close enough, this level curve looks like k lines intersecting at the critical point, forming equal central angles of π/k, something like an asterisk.

Consequences for Curves

Fix a polynomial, fix a point z_0, and imagine the possibilities for the level curves of the modulus of the polynomial. One of these curves passes through z_0. By definition it cannot intersect any other level curve (of a different height), but it can intersect itself at a point where $k > 1$. The Geometric Modulus Principle tells us what any such intersections must look like: an asterisk with $2k$ equal central angles.

A "typical" polynomial, say, one whose roots are randomly chosen, is likely to have just three types of points in its domain: roots, critical points where the second derivative doesn't vanish (i.e., points with $k = 2$), and everything else—points which are neither roots nor critical points (i.e., points with $k = 1$). Figure 3 shows a contour plot of the modulus of one such random polynomial of degree 13. The 13 roots appear as white dots, and the 12 critical points appear as black dots. Each critical point has $k = 2$, so they are "classic" saddle points with two ascent and two descent sectors. The Geometric Modulus Principle says that each of these saddle points has the additional property that the level curve on which it lies crosses itself at perfect right angles. This plot includes only the level curves that pass through a critical point, and provides a compelling portrait of the polynomial.

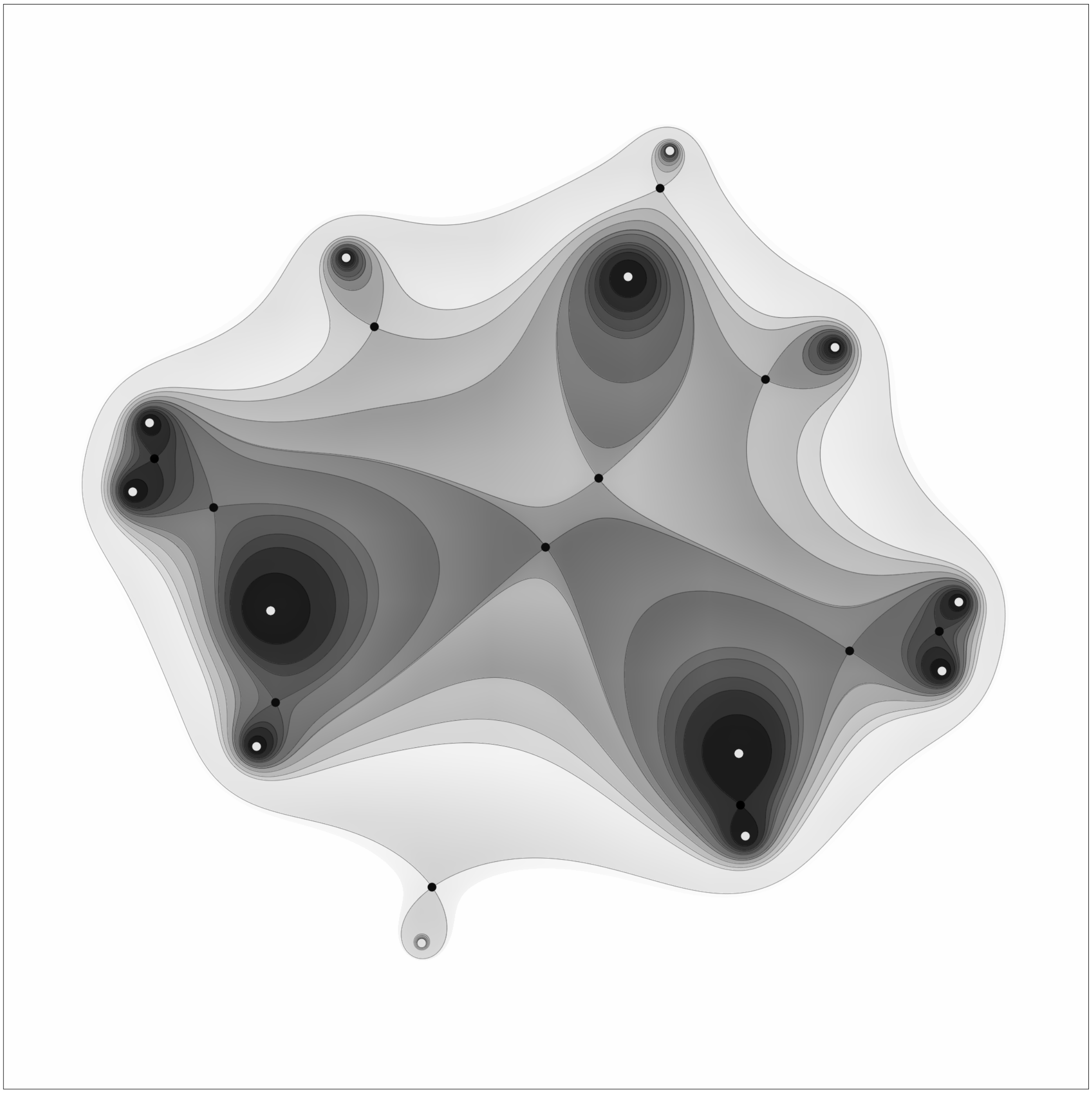

Figure 3: *Modulus plot of a random polynomial of degree 13*

Less typical (but even more beautiful) are polynomials with higher order critical points. Figure 4 shows the modulus plot of the degree-7 polynomial $p(z) = (z^3 - 1)(z^4 - i)$, which has a critical point with $k = 3$ shown near the center. There are six alternating ascent-descent sectors at this point. The plot is superimposed with the modulus plot of its derivative, to create an even richer portrait. The critical points of p are the roots of its derivative, so the plots fit together in an intriguing and natural way. Transparency of the shaded regions in the plot of $p(z)$ permit the eye to glimpse the derivative below. Nontraditional shading in the derivative plot (darker regions are not necessarily "below" lighter regions) is utilized to allow the viewer to take in both images at once. While some information is lost through that choice, the relationship between the two plots is emphasized and enhanced.

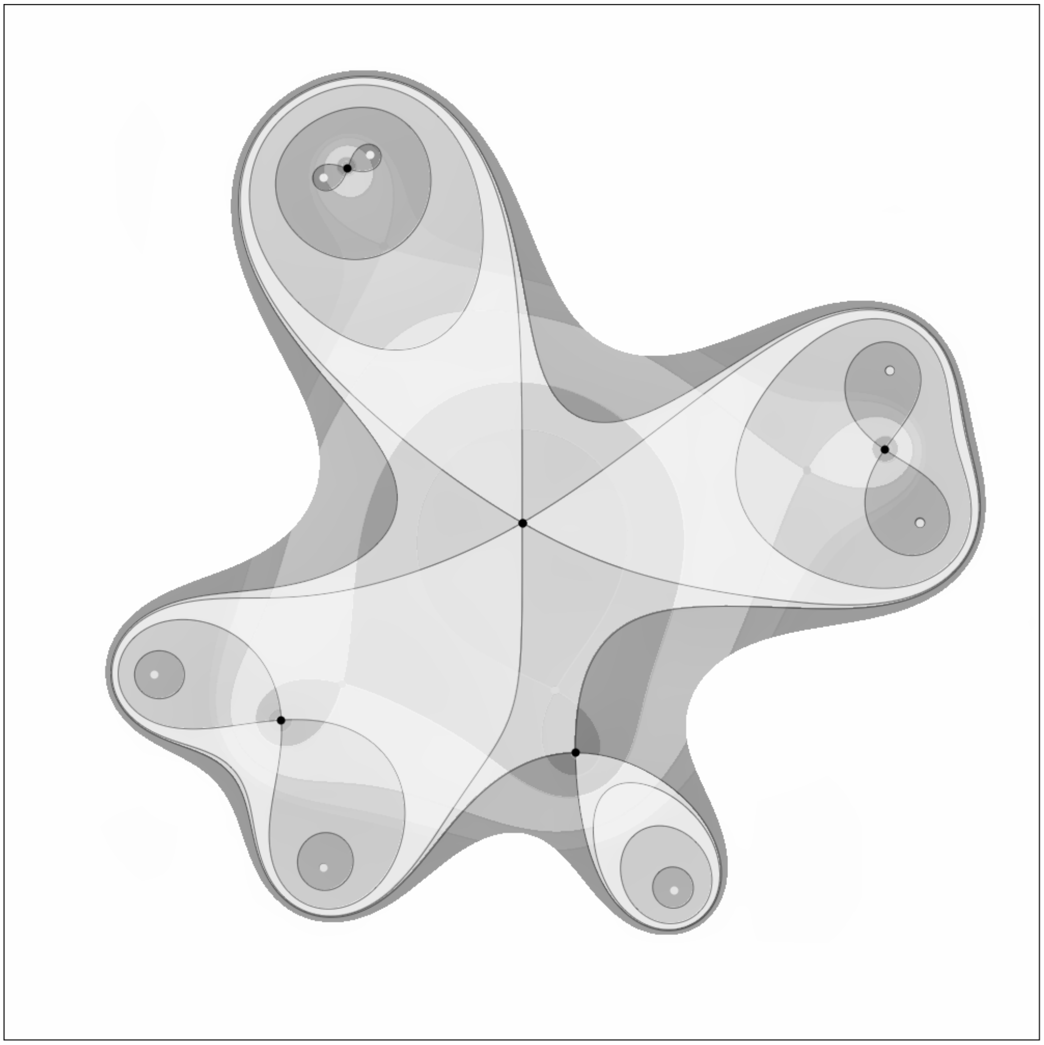

Figure 4: *Modulus plots of* $p(z) = (z^3 - 1)(z^4 - i)$ *and its derivative*

References

[1] B. Kalantari, "A Geometric Modulus Principle for Polynomials," *The American Mathematical Monthly,* Vol. 118, No. 10 (December 2011), pp. 931–935.

[2] B. Kalantari and B. Torrence, "The Fundamental Theorem of Algebra for Artists," *Math Horizons,* Vol. 20, No. 4 (April 2013), pp. 26–29.