Sources of Flow as Sources of Symmetry: Divergence Patterns of Sinusoidal Vector Fields

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Abstract

Here we create two-dimensional black and white symmetry patterns by coloring sources in a sinusoidal vector field in white and sinks in black. We focus on vector fields of the form $\vec{F}(x,y) = \sin(ax)\cos(by)\hat{\imath} + \sin(cy)\cos(dx)\hat{\jmath}$, where $a, b, c, d \in \mathbb{N}$, because of the interesting divergence symmetries such fields produce.

Introduction

Given a vector field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, where $\hat{\imath}$ and $\hat{\jmath}$ are the standard basis vectors of $\mathbb{R}^2$, then the divergence of $\vec{F}$ is the scalar quantity

$$\text{div}\vec{F}(x,y) = \nabla \cdot F = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y).$$

Interpreting $\vec{F}$ as the velocity field of a fluid flow, div$\vec{F}$ measures the tendency of a fluid to diverge from a point. If div$\vec{F}(x_0,y_0) > 0$, then (x_0,y_0) is a source of new fluid; if div$\vec{F}(x_0,y_0) < 0$, then (x_0,y_0) is a sink of old fluid; and if div$\vec{F}(x_0,y_0) = 0$, then $\vec{F}$ is incompressible at (x_0,y_0). Interestingly, it is often difficult to identify the locations of sources (or sinks) in a vector field by simply looking for points at which the flow radiates outward (or inward) in the field. To illustrate, consider the vector field $\vec{F}(x,y) = \sin(2x)\cos(5y)\hat{\imath} + \sin(2y)\cos(5x)\hat{\jmath}$ plotted over the region $[-\pi,\pi] \times [-\pi,\pi] \subset \mathbb{R}^2$ in Figure 1 (left). While the plot clearly indicates a source of new fluid at the origin, it falls short of suggesting the true nature of the intricate divergence pattern revealed when the sources are colored in white and the sinks in black (as in Figure 1 (right)).

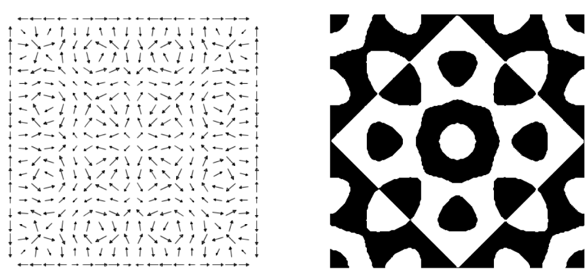

Figure 1: *The vector field* $\vec{F}(x,y) = \sin(2x)\cos(5y)\hat{\imath} + \sin(2y)\cos(5x)\hat{\jmath}$ *(left) with vectors rescaled using MAPLE's "fieldstrength = log" option and divF (right) over* $[-\pi,\pi] \times [-\pi,\pi]$

Divergence patterns of vector fields of the form $\vec{F}(x,y) = \sin(ax)\cos(by)\hat{\imath} + \sin(cy)\cos(dx)\hat{\jmath}$ (where $a, b, c, d \in \mathbb{N}$) display a wide variety of interesting and complex patterns. (See Figure 2.) Since div $\vec{F}(x,y) = a\cos(ax)\cos(by) + c\cos(cy)\cos(dx) = 0$, the periodicity of the cosine function ensures that div$\vec{F}(x,y) =$

$\text{div}\vec{F}\left(x + \frac{2\pi}{gcd(a,d)}k, y + \frac{2\pi}{gcd(b,c)}l\right)$ for all integers k and l, and since cosine is an even function, $\text{div}\vec{F}(-x, y) = \text{div}\vec{F}(x, y) = \text{div}\vec{F}(x, -y)$. Hence all of the divergence patterns we consider have rectangular fundamental regions and reflection axes in both the vertical and horizontal directions. Because $\text{div}\vec{F}(-x, -y) = \text{div}\vec{F}(x, y)$, all such patterns also exhibit rotational symmetry with a $180°$ angle of rotation. If $a = c$ and $b = d$, or if $a = b$ and $c = d$, the divergence patterns also exhibit $90°$ rotational symmetry.

Lines of Incompression

Clearly, the boundary curves separating the black regions from the white regions consist of points at which the divergence is zero. We call such points *points of incompression* in the vector field. Examining Figure 2, we see that some divergence patterns contain vertical lines of incompression (e.g., (i), (iv), and (x)) while others contain horizontal lines of incompression (e.g., (i), (vii), and (x)). Still others contain neither. As the next theorem shows, the existence of such lines is related to the maximum power of 2 dividing the integers a, b, c, and d. Define $\nu_2(n) := \max\{\nu \in \mathbb{N} : 2^\nu | n\}$; $\nu_2(n)$ is called the *2-adic valuation* of the natural number n.

Theorem 1. A vector field of the form $\vec{F}(x, y) = \sin(ax)\cos(by)\hat{\imath} + \sin(cy)\cos(dx)\hat{\jmath}$, with a, b, c, $d \in \mathbb{N}$ such that $a \neq d$ and $b \neq c$ has vertical lines of incompression if and only if $\nu_2(a) = \nu_2(d)$, in which case the lines occur at $x = \frac{1}{\text{gcd}(a,d)}(\frac{\pi}{2} + N\pi)$ for all $N \in \mathbb{Z}$. $\vec{F}$ has horizontal lines of incompression if and only if $\nu_2(b) = \nu_2(c)$, in which case the lines occur at $y = \frac{1}{\text{gcd}(b,c)}(\frac{\pi}{2} + N\pi)$ for all $N \in \mathbb{Z}$.

Proof sketch. Note that vertical lines of incompression occur at x_0 when for all y, $\text{div}\,\vec{F}(x_0, y) = a\cos(ax_0)\cos(by) + c\cos(cy)\cos(dx_0) = 0$, which happens if and only if either

$$\cos(ax_0) = \cos(dx_0) = 0, \text{ or for all } y, \quad \frac{\cos(ax_0)}{\cos(dx_0)} = -\frac{c}{a}\frac{\cos(cy)}{\cos(by)}.$$

Solving this system yields the desired result for vertical lines; the proof for horizontal lines is analogous. $\square$

Remark. If either $a = d$ or $b = c$ then the divergence pattern is a rectangular grid (e.g., Figure 2 (x)).

The existence of diagonal lines of incompression (as exhibited in patterns (ii) and (v) in Figure 2) can also be understood using Theorem 1. To illustrate, consider patterns (i) and (ii) in Figure 3 below. These divergence patterns look suspiciously similar to patterns (ii) and (v) of Figure 2, and indeed, there is a close relationship between them. Defining $\vec{F}_{a,b}(\bar{x}) := a\cos(ax)\cos(by) + a\cos(ay)\cos(bx)$, where $\bar{x} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, and letting M be the matrix $M = \sqrt{2}/2 \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$, then an application of the "product to sum" identity for cosine $\left(\cos(\theta)\cos(\phi) = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}\right)$ quickly yields the identity

$$\text{div}\,\vec{F}_{a,b}(M\bar{x}) = \frac{2a}{b-a}\text{div}\vec{F}_{\frac{b-a}{2},\frac{a+b}{2}}(\bar{x}). \tag{1}$$

If $a = 1$ and $b = 5$, for example, then $\text{div}\vec{F}_{1,5}(M\bar{x}) = \frac{1}{2}\text{div}\vec{F}_{2,3}(\bar{x})$, and because $v_2(1) = v_2(5)$, $\vec{F}_{1,5}$ exhibits horizontal and vertical lines of incompression corresponding to the diagonal lines of incompression exhibited by $\vec{F}_{2,3}$.

Because $A = \frac{b-a}{2}, B = \frac{a+b}{2}$ if and only if $b = A + B, a = B - A$, Identity (1) can be written equivalently as $\text{div}\vec{F}_{A,B}(\bar{x}) = \frac{A}{B-A}\,\text{div}\,\vec{F}_{B-A,A+B}(A\bar{x})$, and we conclude that $\vec{F}_{a,b}$ has diagonal lines of incompression if and only if $\vec{F}_{b-a,a+b}$ has horizontal and vertical lines of incompression. Hence Theorem 1 ensures that $\vec{F}_{a,b}$ has diagonal lines of incompression if and only if $\nu_2(b-a) = \nu_2(a+b)$. As it turns out, it is not possible to find a vector field of the form $\vec{F}_{a,b}$ having neither diagonal nor horizontal and vertical

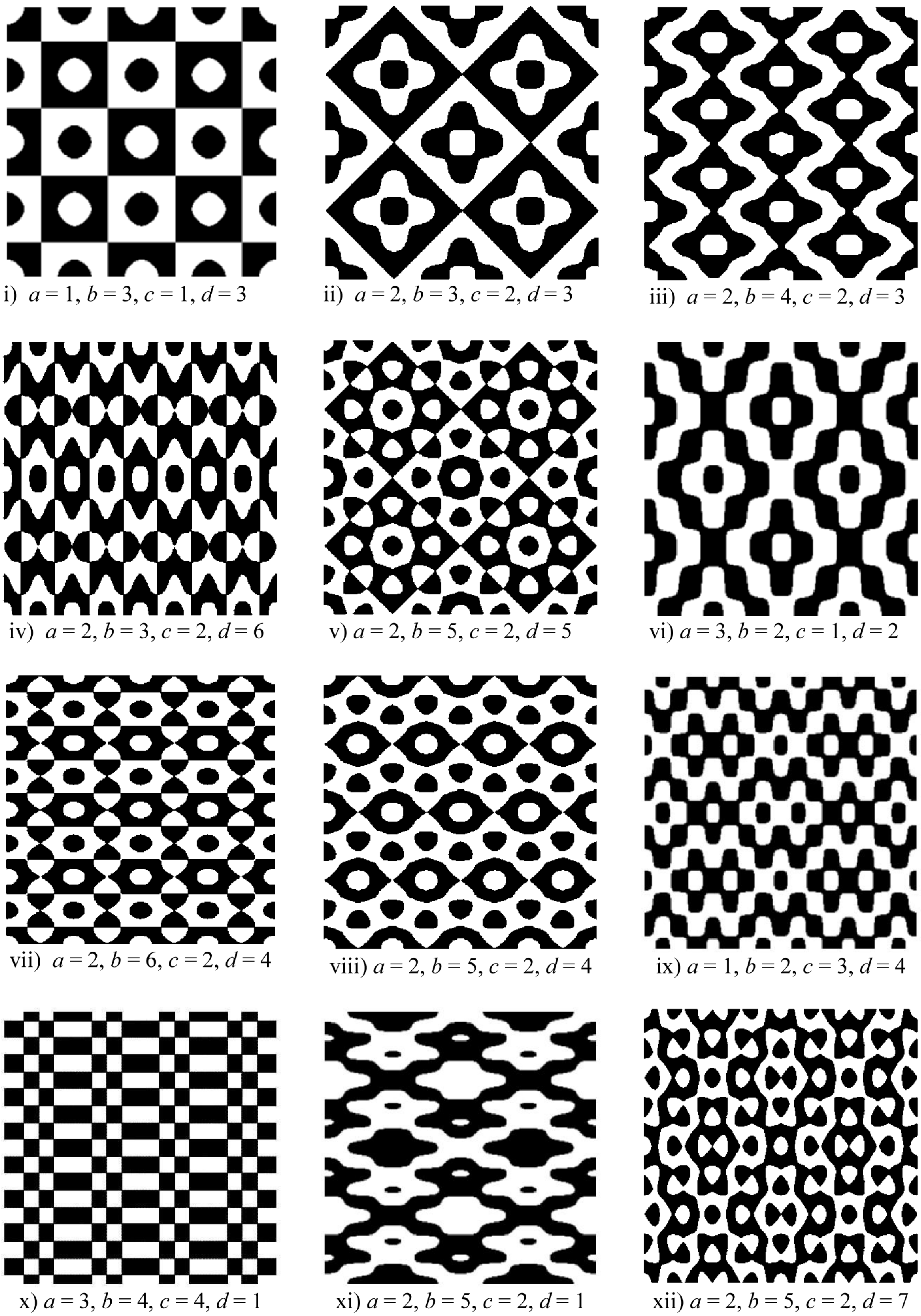

Figure 2: *The divergence patterns for twelve vector fields of the form* $\vec{F}(x, y) = \sin(ax)\cos(by)\hat{\imath} + \sin(cy)\cos(dx)\hat{\jmath}$, *over* $[-2\pi, 2\pi] \times [-2\pi, 2\pi]$.

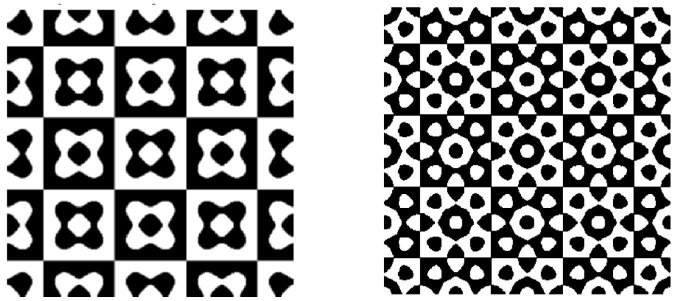

Figure 3 images: i) $a = 1, b = 5, c = 1, d = 5$ ii) $a = 3, b = 7, c = 3, d = 7$

Figure 3 : *The divergence patterns for* $\vec{F}_{1,5}$ *(left) and* $\vec{F}_{3,7}$ *(right) are scaled rotations of the divergence patterns for* $\vec{F}_{2,3}$ *and* $\vec{F}_{2,5}$ *illustrated in Figure 2 ii) and v)*

lines of incompression. If $\vec{F}_{a,b}$ fails to have horizontal and vertical lines of incompression, then Theorem 1 implies $\nu_2(a) \neq \nu_2(b)$. Assuming $\nu_2(a) = k, \nu_2(b) = l$ with $k < l$, it is not difficult to show that $\nu_2(b - a) = \nu_2(a + b) = k$, meaning $\vec{F}_{a,b}$ has diagonal lines of incompression.

Conclusions

Black and white geometric patterns are ubiquitous in contemporary art and design. On wall art, clothing, jewelry, pottery, and even tattoos, one doesn't have to look far to find them. Further exploration of our divergence patterns could include an examination of where such patterns have emerged in the art world. As Figure 4 illustrates, the Ipanema sidewalk in Rio de Janeiro dons a symmetric pattern with notable similarities to our patterns.

Figure 4 : *The Ipanema beach sidewalk in Rio de Janeiro, Brazil exhibits a black and white pattern with symmetries reminiscent of some divergence flow patterns. (Left photo by* `http://www.123rf.com/profile_marchello74`*; right photo by Radek Tezaur.)*

Future work might also include further study into the geometric features of these patterns. It would be interesting to study, for example, the topological connectedness of the black and white regions, and the number of components meeting at a point. More generally, what patterns are possible? Additionally, one could expand our investigation to vector fields other than the sinusoidal fields considered here.