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Hyperbolic Tilings with Truly Hyperbolic Crochet Motifs

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Abstract

Until now, most methods for making a hyperbolic plane from crochet or similar fabrics have fallen into one of two categories. In one type, the work has constant negative curvature but does not naturally lend itself to a polygonal tiling. In the other, polygonal tiles are attached in such a way that the final product approximates a hyperbolic plane on the large scale but does not have truly constant curvature. We show how crochet can be used to create polygonal tiles that have constant negative curvature in themselves and can therefore be joined into a large region of a hyperbolic plane without significant stretching. Formulas from hyperbolic trigonometry are used to show how, in theory, any regular tiling of the hyperbolic plane can be produced in this way.

Introduction: The Mathematics of Hyperbolic Tilings

Regular tilings of the plane have been a longstanding interest of both recreational mathematicians and professional geometers alike. A regular tiling of the plane is a decomposition of the plane into regular polygons such that all the polygons are congruent, each edge of each polygon touches exactly one edge of one other polygon, and the same number of polygons are arranged around each vertex. These tilings are thus completely categorized by the number of sides in the polygon and the number of polygons around a vertex. One notation for representing this is the Schläfli symbol $\{n, k\}$, denoting k regular n-gons arranged around each vertex.

Not every combination of n and k is possible in the Euclidean plane. If we relax the requirement that the geometry be Euclidean, however, we have many more options! We will restrict ourselves to surfaces of constant negative curvature, i.e., hyperbolic planes. The calculation of the internal angles of a Euclidean n-gon relies on the fact that the internal angles of a Euclidean triangle add up to π. However, on the hyperbolic plane sum of the interior angles of a hyperbolic triangle is not constant at all, but is equal to $KA + \pi$, where A is the area of the triangle, and K is the "Gaussian curvature" of the surface. We have $K < 0$ for a hyperbolic plane, and thus depending on the curvature of the surface and the area of the triangle, we can have any possible sum of interior angles between 0 and π. It can be shown from this that each interior angle of a hyperbolic n-gon has measure $KA/n + \pi - (2\pi/n)$.

Going back to our tilings, the sum of the angles around a vertex is still 2π, so the tiling $\{n, k\}$ must have $k(KA + \pi - (2\pi/n)) = 2\pi$, or $nk + (kKA/\pi) = 2(n + k)$. In other words, as long as $nk > 2(n + k)$, then for any curvature we can find an $\{n, k\}$ tiling for hyperbolic n-gons of some area. Or conversely, if we fix the area then there is a surface of some curvature that admits the tiling. (For example, the hexagon in Figure 1 has $n = 6$ and $k = 4$, and interior angles of measure $\pi/2$, from which one can calculate the relationship between the area and the curvature.) There are infinitely many pairs $\{n, k\}$, with $n \geq 3$ and

$k \geq 3$, such that $nk > 2(n+k)$, and thus infinitely many tilings of any given hyperbolic plane. The focus of this paper will be how to construct very close approximations of these tilings (or, to be more accurate, finite portions of them!) out of crocheted polygonal units.

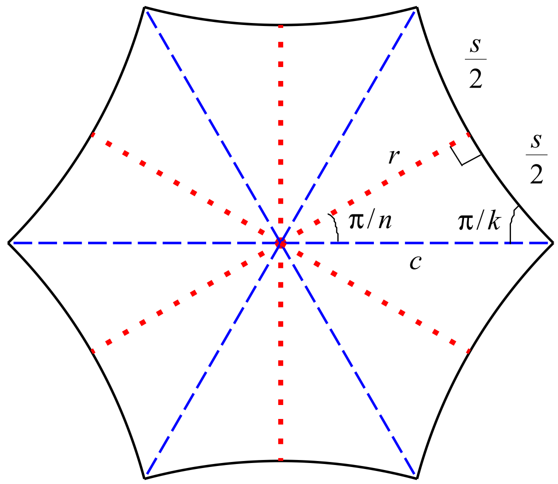

Figure 1 : *A right-angled hexagon (from a $\{6,4\}$ tiling) decomposed into hyperbolic triangles.*

Previous Constructions of Hyperbolic Tilings

There have been many previous attempts to construct such tilings. Henderson and Taimina describe some of them in [4, Appendix B]. For example, one can construct $\{3,7\}$ tilings by putting seven Euclidean equilateral triangles (made out of, e.g., paper) around each vertex. The drawback to these is that the polygonal units themselves are Euclidean. For example, in the case of the $\{3,7\}$ tiling Henderson and Taimina point out [4, p. 371] that this model has $7\pi/3$ radians around each vertex, rather than 2π, and therefore has "cone points" where the model cannot be made smooth. They consider the option of replacing the sides of the triangles with circular arcs such that the vertex angles decrease from $\pi/3$ to $2\pi/7$. Then the vertices will be smooth, but the edges are difficult to join and will resist lying smoothly. (As [7] puts it, the "extra curvature" which would be concentrated at the vertices is instead spread out along the one-dimensional edges.)

Helaman Ferguson [3] got around this last problem by not only giving the polygons curved edges but also making them out of stretchy fleece, which allows the "extra curvature" concentrated along the one-dimensional edges to spread out across the whole two-dimensional surface [7]. He used this technique to make a quilt (based on an earlier poncho) that approximates a $\{5,4\}$ tiling, with four pentagons around each vertex and the interior angles of each pentagon being right angles. However, as Ferguson points out [2], each pentagonal patch is still in fact flat in its resting state.

Another option is taken by Daina Taimina in [6], where she explains how to crochet a portion of a hyperbolic plane and then use a contrasting color of thread to stitch the edges of the tiling. In this case the individual units are not made separately and then joined, but rather demarcated after the fact. Note that the negative curvature is here created by constructing arcs of exponentially increasing length, i.e., rows of exponentially increasing numbers of crochet stitches.

Our technique, unlike these, relies on constructing the tiling units themselves out of fabric with the (approximate) correct curvature. Joining them together is then no longer problematic. Instead, the question becomes: how do we crochet an n-gon of constant negative curvature and the correct angle sum?

Calculating the Correct Shape

To start, we calculate the dimensions of the appropriate hyperbolic n-gon. For the tiling $\{n, k\}$, we decompose the n-gon into triangles as shown in Figure 1. We can use the angle measurements marked on the figure to calculate the inradius r, the circumradius c, and the side length s. We also use the Second Hyperbolic Law of Cosines for a triangle $\triangle ABC$ (see, for example, [5, p. 75]):

$$\cos(\angle C) = -\cos(\angle A)\cos(\angle B) + \sin(\angle A)\sin(\angle B)\cosh(AB)$$

This allows us to compute r, c, and s. For example, in a right-angled hexagon such as the one in Figure 1, $n = 6$, $k = 4$, and if the units are scaled such that $r = 10$, then $c \approx 13.00$ and $s \approx 14.94$. Note that for a Euclidean hexagon with $r = 10$, we would have $c \approx 11.55$ and $s \approx 11.55$.

Crocheting the Polygons

We crochet the tiles using variations of the traditional "granny square" pattern. (See, for example, [1, p. 166].) The inradius will determine the number of rounds of crochet, and is therefore fixed at a convenient number such as 10. We are primarily using double crochet stitches, which we will consider to be 1 unit wide by 2 units high. Thus we have 5 rounds of double stitches. The standard granny square pattern produces a flat surface (zero curvature), so we need to vary the pattern to add the exponentially increasing length required for negative curvature. We do this by adding exponentially spaced increases (extra crochet stitches) on each side until we achieve the desired side length. The left side of Figure 2 shows this for a $\{4, 5\}$ square. The units are again scaled such that $r = 10$, and in this case $c \approx 15.88$ and $s \approx 23.63$. Since in the Euclidean case we would have $s = 20$, we have increased until there are four extra stitches on each side.

The increases give us the desired side length, but we also need to ensure that the circumradius is correct. For this we replace some of the double stitches with treble stitches, which are roughly 3 units high by 1 unit wide. The right side of Figure 2 again shows this for a $\{4, 5\}$ square. Since we want $c \approx 15.88$ and in the Euclidean case we would have $c \approx 14.14$, we replace two double stitches on each side of each diagonal with treble stitches to get roughly the correct length.

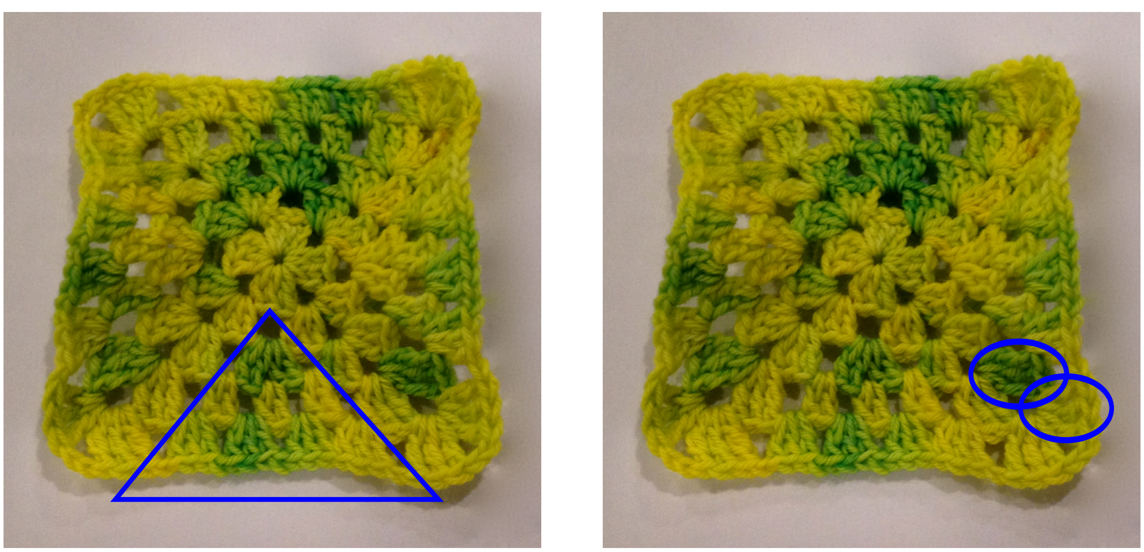

Figure 2 : *A hyperbolic granny square with increases (left) and treble stitches (right) highlighted.*

As a test of the concept, the second author crocheted five $\{4,5\}$ squares and four $\{6,4\}$ hexagons using the principles and measurements above. She joined each set of polygons around a vertex by crocheting them together as she went, with the results shown in Figure 3. Note the ridges where the polygons have been pinched in order to display a hyperbolic surface against a flat background. The next step will be to crochet entire afghans using these constructions.

Figure 3 : *Five hyperbolic squares (left) and four hyperbolic hexagons (right) arranged around a vertex.*

In theory, any hyperbolic tiling could be constructed in this fashion. However, as the interior angles get sharper, it will become more and more difficult to turn the corners. Techniques other than the traditional granny square pattern may be necessary in order to crochet such tilings.

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