A Plane-Filling Curve Using Ammann A5 Tiles

Richard Hassell
WOHA Pte Ltd, Singapore
E-mail: admin@woha.net

Abstract
This paper presents a FASS curve using the Amman A5 aperiodic tile set.

Introduction. Plane-filling, also known as space-filling curves, are a limiting form of fractal. They are usually developed using the recursive substitution methods and are intimately related to tilings with dilational recursive structure. Peano curves, Hilbert Curves and other classic plane-filling curves are based on regular tilings and rep-tiles [3]. Many plane-filling curves have been discovered based on lattices of regular tilings [6], some of which converge on topological disks, while others define more complex boundaries with holes such as the Sierpinski Arrowhead Curve [7, p.74]. Other curves are based on tilings which converge on a fractal boundary through the recursive process, such as the Gosper Curve [2, p. 124]. McKenna has described plane-filling curves using heterogeneous sizes of tiles [5]. Dow [1] has elaborated curves based on square lattices, and has fully described those based on a 2 x 2 grid, with explorations into larger lattices. Ventrella notes the various techniques of making plane-filling curves—Node Replacement Curves, Iterated Function Systems, Branching Fractal Trees, and L-systems. His book and website, which cover many kinds of fractals, defines some curves as "well-behaved" or "FASS"- Plane-Filling, Self-Avoiding, Simple, Self-similar [7, p.46], which the curve described here satisfies. This paper uses a more elaborate notation than Dow and Ventrella, suited for complex tilings. It combines L-system replacement for the curve with marked tile notation for the tile substitution. This explicit marking of the tile is necessary for complex tilings beyond a lattice or regular tilings (where the tile geometry is simply implied), as the line substitution and the tile substitution have to be solved together.

Ammann A5 Plane Filling Curve. Figure 1 shows the tiles used in the Ammann A5 set. [4, p. 556]. The two prototiles are square- and rhombus-based (in bold). Each prototile comprises smaller squares and rhombuses whose arrangement creates notches and bulges along the boundary. These are equivalent to Amman's asymmetric matching rules and key tiles; recursive substitution then generates an aperiodic tiling. To build a FASS curve, we must find a set of starting and ending connection points on the boundary where the substitution tiling vertices coincide:

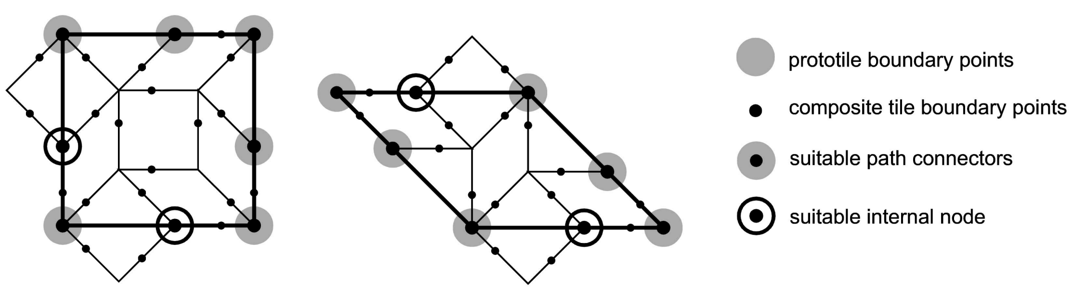

Figure 1: *The Ammann A5 set prototiles, with the composite tile arrangement, and boundary points which will serve as the connection points for the curve.*

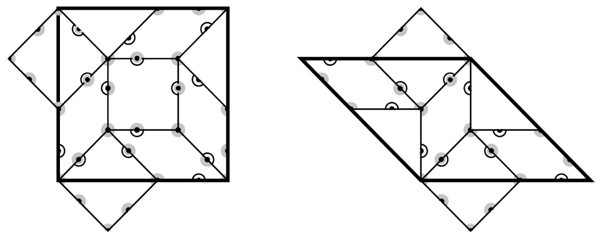

Figure 2: *Points suitable for internal nodes do not coincide when tiled.*

In Figure 2 boundary points suitable for use as "internal nodes" for the composite tile path (white semicircle) are those that do not coincide with other "internal nodes" both within and between composite tiles. These can be used as an intermediate point along the composite tile path without risk of intersection with other intermediate points of adjacent tiles' curves during substitution. The other (gray) boundary points located along the composite tile boundary must be avoided in assembling the paths or self-intersection will occur during substitution. Internal nodes can also be used for start and end points of a curve.

For each marked tile, a composite marked tile is created (see Figure 3), which will be used in the recursive substitution process (Figures 5 and 6). For each tile, a reverse-marked (arrow reversed, colour reversed, but boundary points are not reversed) version is required to solve the puzzle. The boundary points, start point, end point, edge colour and arrow direction must all be matched between the marked prototile and its corresponding composite prototile. The composite path only touches the boundary points at the start-point, end-point and internal nodes:

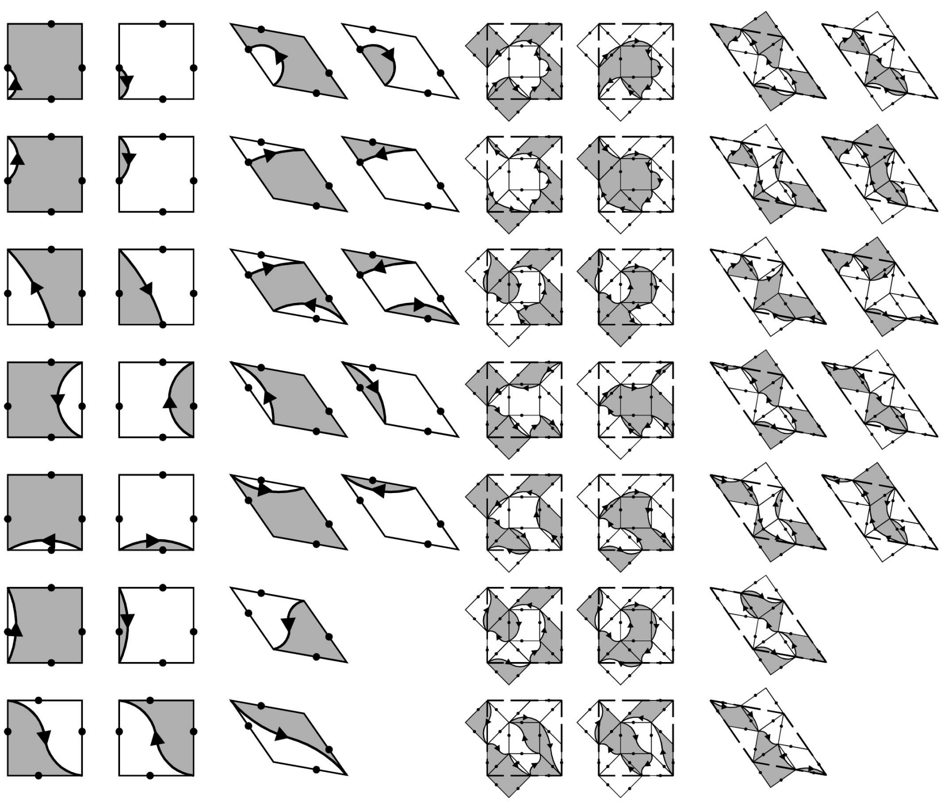

Figure 3: *The Ammann A5 set has 14 square Marked Prototiles and 12 rhombus Marked Prototiles (left). The composite tiles are shown (right).*

Not all composite tiles use all the marked tiles, but all marked tiles are needed to solve at least one composite tile. The last two rhombus tiles have reverse-marking symmetry (are the same as the original tile after reverse-marking) so do not have a reversed version, while the last square tile does not, as it has asymmetric boundary points.The rhombus composite tile has a waist with a single rhombus crossing between the obtuse vertices. In several of the marked composite rhombus, two paths are needed to join the start- and end-points of the composite curve whilst maintaining the direction of the path and the edge colouring of the tile (Figure 4).

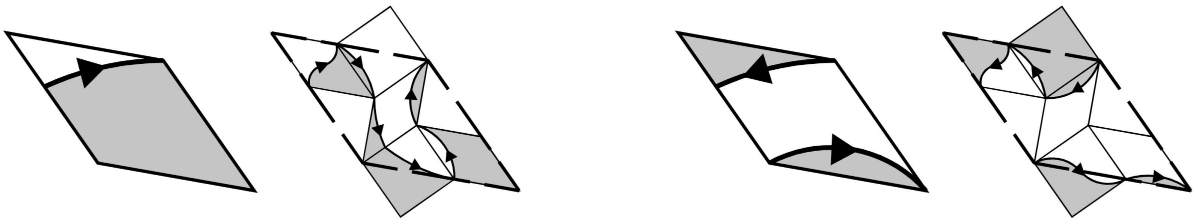

Figure 4: *Left: The rhombus prototile needs a double path within its central sub-rhombus to stay "well-behaved." Right: The two disconnected paths will each recursively evolve.*

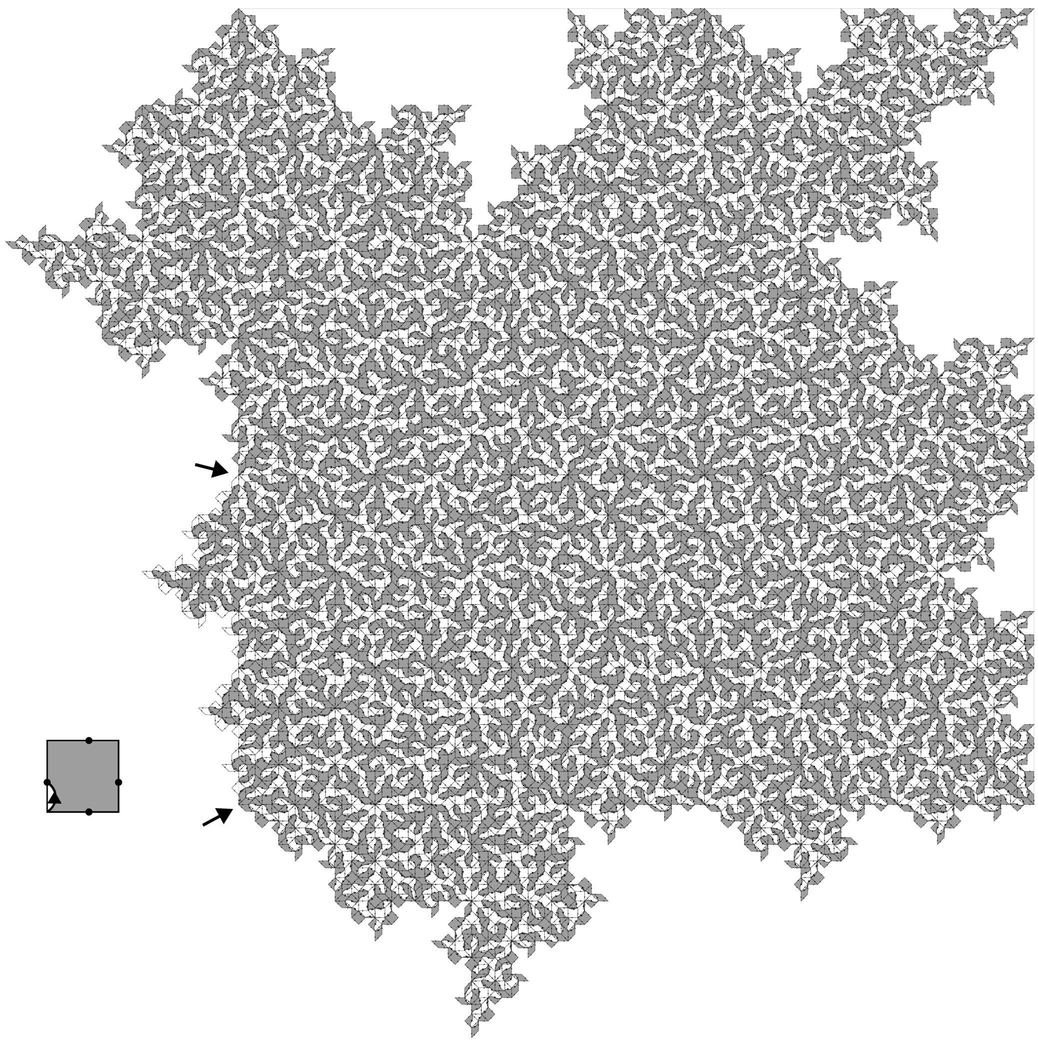

Figure 5: *The fourth stage of substitution on the first square tile, shown at left at reduced scale. Start-point is at bottom left, end-point on left side, edge colouring and arrow direction are maintained.*

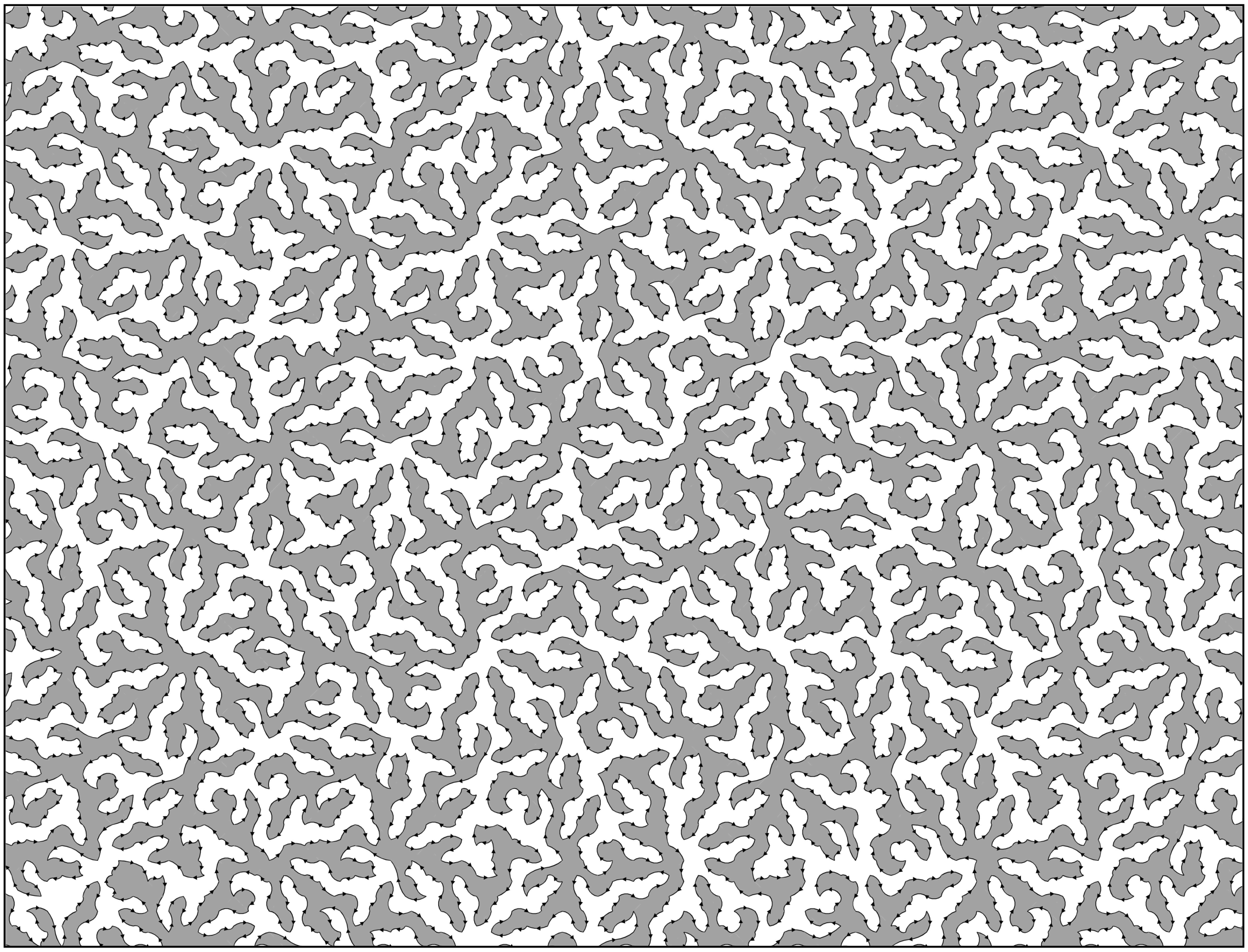

Figure 6: *A portion of the Amman A5 Plane-Filling Curve.*

Conclusion Plane-filling curves exist for complex tilings. When tilings beyond a simple regular tiling lattice are used as the organizing geometry for a plane-filling curve, the markings that govern the tile substitution need to be given equal prominence to the curve substitution; plane-filling curves can be studied as marked tilings. The Amman A5 plane-filling curve overlays a linear fractal construction on an aperiodic spatial construction. The double recursive complexity give this pattern its beauty.

References

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