Capturing Eight-Color Double-Torus Maps

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Abstract

The extensions of the Four-Color Map Theorem to surfaces of higher genus are difficult to visualize. For instance, without a physical model of a torus map with seven pairwise adjacent countries, it is hard to imagine that there are maps on a torus that require seven colors to color all the countries so that no adjacent countries are the same color. Making maps with large numbers of pairwise adjacent countries is a delightful artistic challenge that lends itself to a variety of media. Here, we present a scheme to produce a map that requires the maximum eight colors on a two-holed torus and show two artworks made with that scheme: a painted ceramic tea set and a bead crochet pendant.

Eight-Color Maps on the Double Torus

For Gathering for Gardner 8 in 2008, Carolyn Yackel made a pair of hyperbolic pants that could be folded into a map on a double torus with eight countries, each of which touches all the others. Her map, a variation of the single-color knitted pattern in [3], illustrates the fact that up to eight colors are required to color a map on the two-holed torus in such a way that no adjacent countries are the same color (see [4]). As a mathematical artist and an avid collector of seven-color maps on the one-holed torus, I was eager to adapt this pattern to other media, but found that Yackel's scheme did not lend itself naturally to non-hyperbolic surfaces. Like the original knitting pattern, her map used the traditional gluing of an octagon to form a two-holed torus shown in Figure 1.

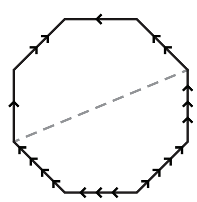

Figure 1: *The standard gluing of an octagon to form a two-holed torus.*

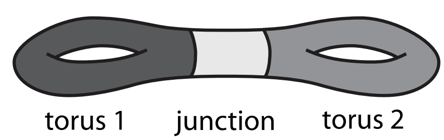

Figure 2: *A two-holed torus decomposed into two punctured tori. The marked diagonal in Figure 1 corresponds to the junction between the two tori in Figure 2.*

While it is not too hard to see that the result will be a two-holed torus by considering what happens on each side of the dotted diagonal, it is challenging to visualize how the octagon's edges wrap around a more conventionally formed two-holed torus like that in Figure 2. Is there a map scheme that lends itself more readily to different shapes of double tori?

To formulate a nice arrangement of colors, we start by considering the countries as polygons, with the understanding that in a physical realization their edges may not be straight lines or even geodesics—or on the other hand, some of their edges may be collinear. In a seven-color map on a torus, the regions are all hexagons (because they border six other countries), and there are three hexagons to each vertex. Note that this agrees with the torus's Euler characteristic of 0: since there are 7 hexagons with 42/2 = 21 edges and 42/3 = 14 vertices, $V - E + F = 14 - 21 + 7 = 0$. For a map on the double torus with eight countries, there will be 8 heptagons with 56/2 = 28 edges. Since the Euler characteristic of the two-holed torus is –2, we have $V - E + F = V - 28 + 8 = -2$, and we conclude that $V = 18$. The total number of vertices in the heptagons counting multiplicities is 56; we can thus reach the required 18 vertices in our map by having two vertices with four heptagons around them and the rest with three. Symmetry suggests that we divide our heptagons into two sets of four and attach each set to one of the degree-four vertices.

With considerable experimentation, we arrive at the coloring scheme in Figure 3. The top two diagrams show each cluster of four heptagonal countries arranged into an octagonal region, with the countries they are attached to shown around the octagon. Each cluster is identified along the vertical and horizontal edges (as indicated by the arrows) into a punctured torus, and the diagonal edges form the attachments between the two punctured tori. Looking at the two octagons, we see that each country touches the junction between the tori in two different places. In these streamlined diagrams, each country appears pentagonal rather than heptagonal because three of the edges are drawn along the same diagonal.

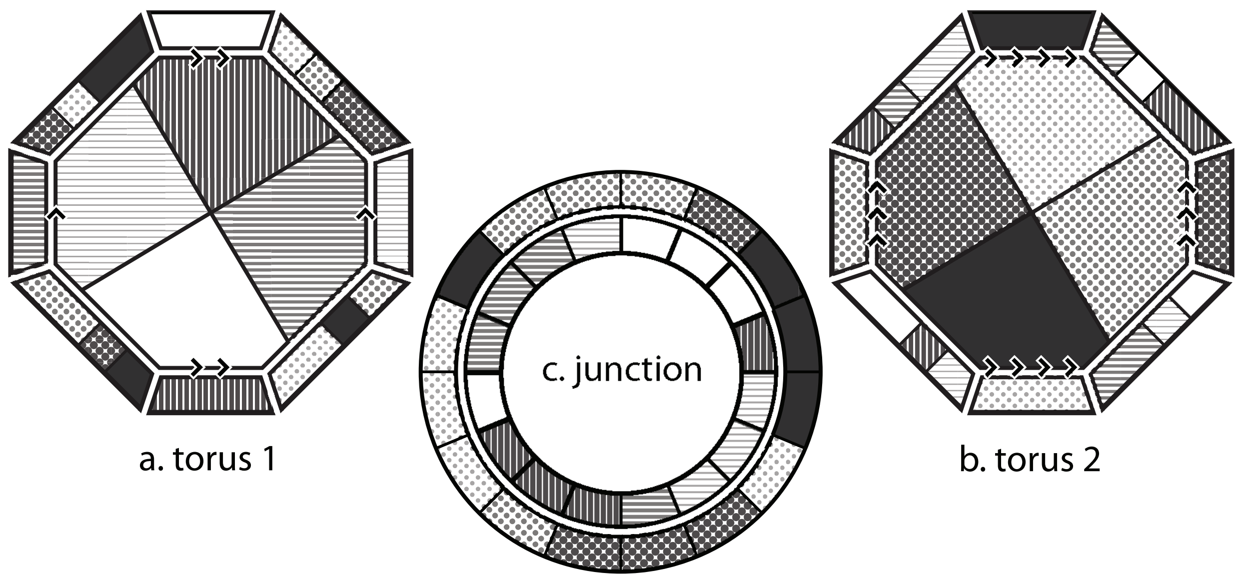

Figure 3: *A scheme for producing an eight-color map on the two-holed torus. Diagrams a and b show each punctured torus surrounded by eight strips showing which colors are glued to the boundaries. The vertical and horizontal edges of each octagon are glued together to form the punctured torus, and the diagonal edges are the boundary of the puncture. Diagram c shows the color sequence around the puncture; the colors from torus 1 are on the inner ring and those from torus 2 are on the outer ring.*

The diagram at the bottom of Figure 3 shows how the countries connect around the junction between the two punctured tori, with torus 1 on the inside and torus 2 on the outside. This region is divided into sixteen segments, each of which contains one of the sixteen possible pairs of colors with one color from each of the two tori. Notice that within this band, each country touches the other seven. This gives the artist applying the scheme some design flexibility; as long as the region between the two tori follows the pattern in this junction diagram, any arrangement of the four countries on each punctured torus will do.

A Map for the Kitchen

Figure 4 shows *Tea for Eight*, a ceramic tea set from the Joint Mathematics Meetings in 2011. The teapot and the teacup are both tori, and they stack to form a two-holed torus, making the set an ideal medium for a double-torus map. I formed the design, painted and fired on prefabricated ceramics, by dividing the circular rim between the pieces into sixteen equal segments, applying the colors following the pattern in the middle of Figure 3, and continuing the colors to the rest of the tea set in an aesthetically appealing way. Unfortunately, it is impossible to extend the two four-color maps on the outside of the teapot and the teacup into seven-color maps, but the set does open up to reveal a six-color map on each torus; the hidden interior region in each torus is simply divided in half so that each half touches all four colors on the rim.

Figure 4: Tea for Eight*, a painted ceramic tea set from the Joint Mathematics Meetings Exhibition of Mathematical Art in January, 2011. It can be used to make and serve tea.*

Because of the spout, the teapot is only a one-holed torus when its lid is closed; otherwise, the channel between the spout and the open top make it a double torus. As the interior of the teapot is unpainted and the exterior of the spout is red, there is technically an extraneous white country that only touches the red.

A Map for the Jewelry Box

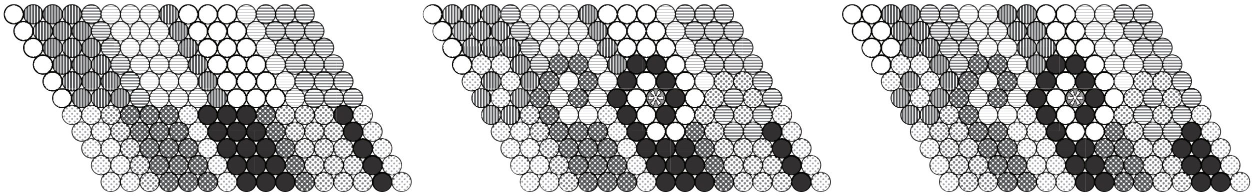

Figure 5: *Translating the junction from Figure 3 into a bead crochet design. The chart on the left is the starting point, and the design was embellished and refined in the following charts. The bead marked with an asterisk in the second and third charts is an ornamental silver bead that is not part of any country.*

There are a number of patterns for seven-color maps on bead crochet bracelets in [2] and [5]. While the conventional form for bead crochet rope jewelry is a torus, it is possible to make a two-holed torus by grafting two narrower segments of bead crochet rope to a thicker central portion. Designing this central portion with a circumference of sixteen beads makes it feasible to apply the coloring from Figure 3. Figure 5 shows a series of bead crochet charts I made while designing the center of the bead crochet pendant *Eight-Color 8* in Figure 6. In the process of embellishing the design (as shown from left to right), the initial pattern using the sixteen diagonal lines of beads is incrementally deformed without changing the country adjacencies—except for the addition of one ornamental silver bead (marked with an asterisk) that can be counted as part of either of the two countries that enfold it. The interlocking pattern on the thinner sections of bead crochet rope uses the design principles described in [1] and [2].

Figure 6: Eight-Color 8*, a bead crochet pendant in the art exhibition for Bridges 2014.*

Acknowledgements

My first painted torus map was a seven-color coffee mug I adapted from a hand-sculpted mug designed by David Lehavi and made by Tamar Ziegler; without their inspiration I would not have started down the torus-map path. This enthusiasm for seven-color maps lead to my collaboration with Sophie Sommer and Ellie Baker, who introduced me to bead crochet and an endless treasure trove of mathematical discovery. I am also indebted to Carolyn Yackel and sarah-marie belcastro for introducing me to the eight-color pants in particular and to mathematical fiber arts in general.

References

[1] E. Baker and S. Goldstine, *Bead Crochet Bracelets: What Would Escher Do?,* Bridges Towson: Mathematics, Music, Art, Architecture, Culture: Conference Proceedings 2012, Tessellations Publishing, July 2012, pp. 567–572.

[2] E. Baker and S. Goldstine, *Crafting Conundrums: Puzzles and Patterns for the Bead Crochet Artist*, A.K Peters/CRC Press, to appear July 2014.

[3] s-m. belcastro and C. Yackel, *Stop Those Pants!*, Chapter 10 of *Making Mathematics with Needlework: Ten Papers and Ten Projects*, s-m. belcastro and C. Yackel editors, A.K. Peters, December 2007, pp. 155–177.

[4] D. Gay. *Explorations in Topology*. Academic Press, Burlington, MA, 2006.

[5] S. Goldstine, S. Sommer, and E. Baker, *Beading the Seven-Color Map Theorem*, Math Horizons, Sept 2013, pp. 22—24.