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Three Games Involving Bungus and Prime Numbers

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Abstract

Inspired by recent spectacular breakthroughs in number theory, we introduce three classroom games involving prime and Bungus numbers. Although prime numbers have captured the imagination of mathematicians for millennia, some of their most intuitive fundamental properties have proven to be very elusive to formal demonstration until up to this past year, when they were finally proven for the first time. We think that our games, if accompanied by the proper teacher guidance, will inspire students to deepen their knowledge of number theory, and to pursue further exploration of current prime numbers mathematical research and/or game designing.

Some Recent Spectacular Advances on the Gaps of Primes Numbers

Main Definitions. A *Prime Number* is an integer bigger than *1* that has only *1* and itself as positive integer divisors. *Twin Pairs of Prime Numbers* are pairs of primes that differ by two, such as *3* and *5*, or *11* and *13*. More generally, the *Gap* of a *Pair of Prime Numbers* is equal to the absolute value of their difference. For example, the gap between *17* and *11* is *6*. Moreover we shall see that all the twin prime pairs are numbers of type $6n-1$ or $6n+1$ for some integer n (e.g., $11=2\cdot 6 -1$, and $13 = 2\cdot 6 +1$).

Recent Advances. A very famous and old unsolved question in the theory of prime numbers asks if there are infinitely many pairs of twin prime numbers, or, more generally, if there are infinitely many pairs of primes within a fixed gap x. Intuitively, the obvious answer is Yes, but this has proven to be extremely difficult to demonstrate rigorously. Indeed, the first rigorous proof of a gap conjecture was only given last year, for $x=70$ *million*, [3], [4]. Current research in this area has reduced the value of the gap x to 600.

Prime Numbers and Bungus Numbers

Bungus Numbers. We will call the numbers of the form $6n\pm 1$ *Bungus Numbers* since they seem to have been explored by P. Bungus in the 16th century.

Prime Numbers and Bungus Numbers. It has been known for long time that, with the exception of the primes *2* and *3*, any prime number is of type $6n+1$ or $6n-1$, where n is a nonnegative integer. To prove it, take a prime p (different from *2* and *3*) and look at its congruence class k *modulo 6* (or, in short, *mod 6*). This means that $p=6\cdot s +k$, with s and k integers, and k between *0* and *5*. The only possible values for k are *0, 1, 2, 3, 4,* or *5*. If $k = 0$, *2* or *4*, p is even since it is divisible by *2*; similarly if $k=3$, then p is divisible by *3*. So p can only be prime if p is congruent to *1* or *5 mod 6*; since $5= 6\cdot 0 +5$, and $-1= 6\cdot(-1) +5$, we can replace *5* by *-1*. We will call the numbers of type $6n\pm 1$ *Bungus Numbers* in what follows; so any Bungus numbers *mod 6* can be identified with either +1 or -1; we will use this in our first game.
"*Sieving.*" The new exciting prime number gap results are based, among other things, on an old method called "sieving" used to filter out primes. The idea is to first cross out any number on the list of Bungus

numbers that is divisible by three. Next cross out the multiples of five, then of 7, and so on. The numbers that survive this crossing-out process are the primes. We will use this idea in our BB game, in where non-prime Bungus numbers will play a prominent role.

The Bungus Area Guessing Game (BAGG)

The first game we will present, the BAGG in short, can be played by elementary school students, with calculations omitted or simplified, and with suitable area models provided by the teacher, see below. We first review a few basic concepts.

Multiplication of Bungus Numbers. Given two Bungus numbers (that is, two numbers of type $6n\pm1$, where n is a nonnegative integer), their product is a Bungus number since $(\pm1)\cdot(\pm1) = \pm1$. As a consequence, any power and product of powers of Bungus numbers is a Bungus number. For example $7^2 \cdot 11^3 = (6\cdot8 +1)(6\cdot222 -1) = 6\cdot(8\cdot1332 - 8\cdot1 + 222\cdot1) -1 = 6\cdot10{,}870 -1$.

Multiplication of Bungus Numbers mod 6. Any Bungus number *mod 6* can be identified with either $+1$ or -1. Therefore, the algebra of multiplying Bungus numbers *mod 6* can be identified with the multiplication table on the two elements +1 and –1 where multiplication is given by:

$(+1)\cdot(+1) = (+1), (-1)\cdot(-1) = (+1), (-1)\cdot(+1) = (-1)$, and $(+1)\cdot(-1) = (-1)$.

The mod 6 Bungus Kernel of an Integer. Now let's decompose a fixed positive integer w into the product of its prime integer factors, that is, $w=2^{a(2)} \cdot 3^{a(3)} \cdot (B1)^{b(1)} \cdot .... \cdot (Bq)^{b(q)}$, with $B1, ..., Bq$ different prime Bungus numbers respectively raised to the exponents $b(1),..., b(q)$. We will call the product $(B1)^{b(1)}..(Bq)^{b(q)}$ the *Bungus Kernel of w*. The *mod 6* Bungus Kernel of w is the reduction *mod 6* of the Bungus Kernel of w. For example, let $w= 23{,}400= 2^3\cdot3^2\cdot5^2\cdot13$. Then $a(2)=3$, $a(3)=2$, so the *Bungus Kernel* of $23{,}400$ is $23{,}400 / (2^3\cdot3^2) = 325= 25\cdot13$, and its *mod 6* Bungus Kernel is $+1$ as $325= 6\cdot54 +1$. Teachers can illustrate geometrically the above concepts in the following way.

A Geometric Interpretation. Let $w=2^{a(2)} \cdot 3^{a(3)} \cdot (B1)^{b(1)} \cdot .... \cdot (Bq)^{b(q)}$. First construct a rectangle S with sides $a= (B1)^{b(1)}$ and $b= (B2)^{b(2)} \cdot .... \cdot (Bq)^{b(q)}$, and hence having (Bungus) area equal to $(B1)^{b(1)} \cdot .... \cdot (Bq)^{b(q)}$. By putting together $2^{a(2)} 3^{a(3)}$ copies of S, we can build a rectangle R with area w. The area of S *mod* 6 (equal to the *mod 6* Bungus Kernel) will be called the *mod 6* Bungus Area of w. For example, if $w= 4\cdot25\cdot13$, the rectangle R with sides $(4\cdot25)$ and 13 is built up by four subrectangles S each with sides 25 and 13 and area $25\cdot13$. So the Bungus area of w is $325= 25\cdot13$, and the *mod 6* Bungus Area of w is 1.

The BAGG. There are two players, BUNG1 and BUNG2. The two players challenge each other, asking of the other player in turn "What is the Bungus area of *w*? What is the *mod 6* Bungus area of *w*?" The winner is the player who gives the highest number of correct answers to the other player's questions.

Example. What is the Bungus area of $w=222{,}156$? And the *mod 6* Bungus area of $w=222{,}156$? Solution: $222{,}156 = 2^2 \cdot 3^3 \cdot 11^2\cdot17$. The Bungus area of $222{,}156$ is then $2057=11^2\cdot17$, and its *mod 6* Bungus area is -1 since 11 and 17 are congruent to -1 *mod* 6, and so $11^2\cdot17 = (mod\ 6) = (-1)^2\cdot(-1) = -1$.

The BungusBoard Game (BBG)

This game is based on the fact that the only non-prime Bungus numbers between 1 and 72 are ***25 = 6·4 +1 = 5·5***, ***35 = 6·6 -1= 5·7***, ***49 = 6·8 +1= 7·7***, ***55 = 6·9 +1 = 5·11***, and ***65 = 6·11 -1 = 5·13***.

The BBG. The (pink) BungusBoard (BB) has 72 pink squares, and the squares are numbered sequentially from the left to the right and from the bottom to the top as shown in Figure 1. For each of the special numbers ***25***, ***35***, ***49***, ***55***, or ***65***, we constructed a box with area equal to the number itself; call **R(*25*)** the box with area 25 and boundary a solid line; **R(*35*)** the box with area 35 and a dotted boundary line, **R(*49*)** the box with area 49 and boundary line made of small squares; **R(*55*)** the box with area 55 and boundary line made of small asterisks; and **R(*65*)** the box with area 65. These boxes are shown in Figure 1. For example, **R(*65*)** is the blue box at the bottom of the BB whose base extends *13* squares starting from the square number *1*, and its height is *5* squares; **R(*49*)** is the brown box whose base extends *7* squares starting from the square number *1*, and its height is *7* squares, and so on.

The BBG Set-Up. The game is played by two players: BUNG1 and BUNG2. Each of the players starts the game with 6 *Bungus Tokens* and five *Super-6 Bungus Tokens* (each in the denomination of 6 Bungus tokens) in his/her possession. The winner is the player who runs out of Bungus tokens first. The game is subdivided into phases, and each phase has three parts. In the first one player, which we will call the *Roller*, throws a six-sided die (the throw of a 6 is void; the player throws the die again); in the second the other player, which we will call the *Chooser*, chooses between a) or b) (see below for details on a) and b)); and in the third the rules i) and ii) are applied (see below for details on i) and ii)). The two players BUNG1 and BUNG2 alternate being the Roller and the Chooser in subsequent phases.

A Phase of the BBG. Say the Roller's throw is *n*; then the Roller must add *6n* of his Bungus tokens to the pink BB, one per square starting from the first empty square, in increasing order. If the Roller does not own *6n* tokens, he must put on the BB the highest number equal to a multiple of *6* of tokens he/she owns. In this way, with the die's throw, the board reaches a number multiple of 6. It is now the Chooser's turn to play. He/she can only

a) Add a token on the lowest-numbered empty square of the pink BB; or

b) Take the token on the highest-numbered square of the pink BB.

With either a) or b), the board reaches a Bungus number: ***N***. At this point only two cases can occur:

i) ***N*** is a prime. In this case the Roller must take the tokens on the BB that are on the highest row;

ii) ***N*** = ***25***, ***35***, ***49***, ***55***, or ***65***; players now yell "Bung!" In this case, the ***N*** BB tokens must be redistributed over **R(*N*)**, the box corresponding to ***N***. After this redistribution, the Roller must take the tokens on the white part of **R(*N*)**. E.g., if ***N*** = ***65***, **R(*N*)** = **R(*65*)**, so the Roller must take *35* tokens. (Alternatively, for a longer version of the game, the Roller must take the tokens on the white part of **R(*N*)**, while the Chooser takes the tokens on the pink part of **R(*N*)**.)

Example. If in the first phase of the game, BUNG1's throw is a *5*, he/she puts *30* tokens down; if BUNG2 decides to take a token, the board reaches *29*, and so BUNG1 must take back the *5* tokens on *25-29*.

Alternative Way to Play the BBG. As an alternative, shorter, and less conceptual way to play the game, if after a die's throw the board reaches one of the squares labelled by the numbers ***24***, ***36***, ***48***, ***54***, and ***66***, then the other player must follow the Game Instructions for each of those squares given in Figure 1.

The Bungus Twin Primes Game (BTPG)

The BTPG. This game is based on the fact that there only *7* twin pairs of prime Bungus numbers between *1* and *72*, namely: *5,7; 11,13; 17,19; 29, 31; 41, 43; 51, 53; and 59, 61*. The BungusDeck consists of 22 cards, each labelled by a Bungus number between *1* and *72*. There are two players, BUNG1 and BUNG2. To begin, after the cards are shuffled, the deck is placed on the table with the cards facing down. Then the two players pick three cards each from the top of the deck; they look at them, without showing them to the other player. One of the two players, the *Thrower*, plays one of his three cards by placing it face-up on the table. The other player, the *Respondent*, must now put down on the table the card, among his three, that is closest in number to the Thrower's card. Now, if the two cards that have been played form a twin pair of primes, the players now yell "Bungus Twin!" and the Thrower keeps both cards as *Bungus Treasure*. Otherwise, the two cards on the table are placed face down at the bottom of the deck. Each of the two players picks now a card from the top of the deck, and the process starts again with a new throw. The players alternate being the Thrower and the Respondent. The game is over when all the twin pairs of prime numbers have been collected as Bungus Treasure; the winner is the player whose Bungus treasure contains the majority of twin prime pairs. For example, assume that the Thrower holds 7, 21, and 29, while the Respondent 5, 23, and 43. If the Thrower plays 7, then the Respondent must play 5; so the Thrower collects 7 and 5 as his/her Bungus Treasure. If the Thrower plays 29, then the Respondent must plays 23; so 7 and 23 go to the bottom of the deck.

Game Design Workshops: A Fun Way to Do Mathematical Research!

The exploration of generalized versions of our games can be part of an in class game-designing workshops; see also [1] and [2] for math games. We suggest using the following guidelines.

Extensions of the BAGG. The BAGG could include volumes and hypervolumes, so that each Bungus factor $(Br)^{b(r)}$ could represent the length of a dimension. In addition, fun rules for developing a multiplication table of Bungus kernels *mod 6* can be easily deduced from the above Sections.
BBG Workshops. The number of squares in the BB is increased so there is no need to void the throw 6; the number of Bungus tokens is raised accordingly. As before, the game is divided into three phases, and proceeds in a way similar to what outlined above. As part of the workshop the teacher could ask the students to custom (re)design the boxes **R(*N*)** for the nonprime Bungus numbers to optimize the game.

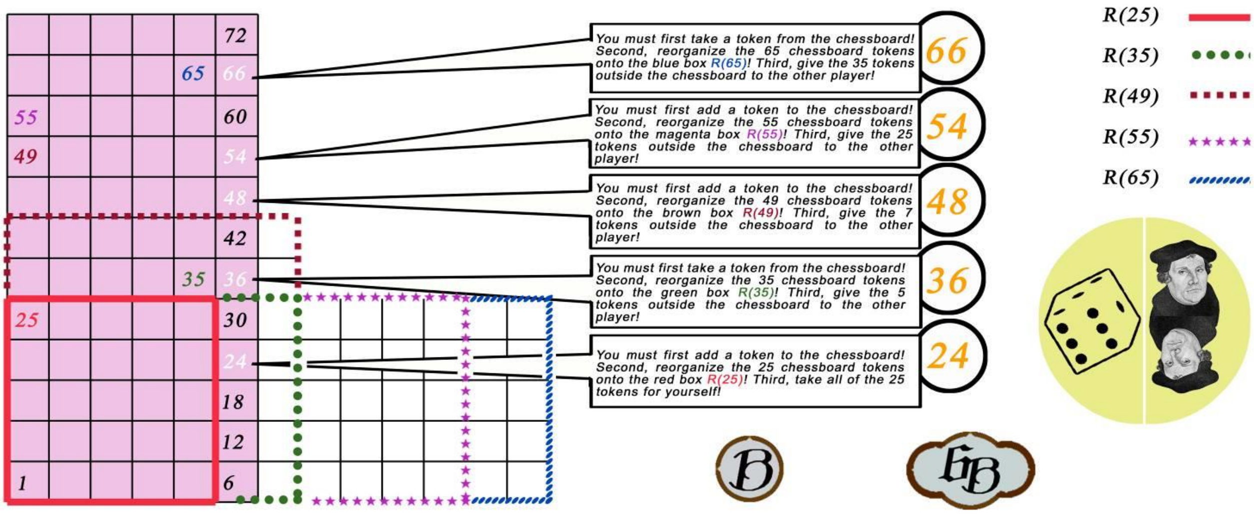

Figure 1: *The BungusBoard (left) with **R(25)**, **R(35)**, **R(49)**, **R(55)**, **R(65)**; Game Instructions (middle), a Dice-Turn Indicator (right), a Bungus Token and a Super-6 Bungus Token (bottom).*

The Bungus Twin Primes Game. Alternatively, the BungusDecks can contain the 8 twin pairs of prime Bungus numbers between *1* and *100*; this time the game can also end in a draw.

Students' Feedback: the University of Colorado Undergraduate Math Club

Feedback. The students played the games for about an hour. They seemed to have a lot of fun, and they had a lot of suggestions, many of which we have incorporated in our paper such as such as the name BungusBoard for the BBG, or the Super-6 Bungus Token (in the denomination of 6 Bungus tokens). They also said they liked the BungusBoard to be very large, and that a fun variant of the BBG is to play without the BB and keep track of the number of tokens in your head.
We also found out that the word "Bung" makes people laugh; this helps make the game a fun experience.

References

[1] J. H. Conway, *On Numbers and Games*. Second edition. *A K Peters, Ltd., Natick, MA,* 2001.

[2] M. Gardner, *Martin Gardner's Mathematical Games*. The Entire Collection of his Scientific American Columns. MAA Spectrum. *Mathematical Association of America, Washington, DC,* 2005.

[3] E. Klarreich, *Unheralded Mathematician Bridges the Prime Gap*, Quanta Magazine, May 2013, https://www.simonsfoundation.org/quanta/20130519-unheralded-mathematician-bridges-the-prime-gap/.

[4] E. Klarreich, *Together and Alone, Closing the Prime Gap*, Quanta Magazine, Nov. 2013, https://www.simonsfoundation.org/quanta/20131119-together-and-alone-closing-the-prime-gap/.