Three Color “2 : 1 : 1” Designs

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Abstract

We analyze an extension of symmetric colored patterns where there are three colors, one of which occurs twice as much as the other two, but where we still have full symmetry between the colors: There is at least one symmetry that fixes the larger color while interchanging the two smaller colors, and at least one symmetry that interchanges the larger color with the union of the two smaller colors. Such “2:1:1” designs occur in traditional patterns in central, frieze, and wallpaper designs. Here we classify the 33 possible types of 2:1:1 frieze designs, and the 4 types of 2:1:1 central designs, we provide constructed examples of all of them, and describing the ways in which these designs arise in the arts of various cultures. An associated web page gives photos of many examples.

1. Introduction & Terminology

To someone interesting in symmetries and patterns, a “color pattern” (or “color design”) is not simply a pattern that has been colored in some fashion. We insist that the coloring must be done “in coordination with” the symmetries of the design. Normally, this means that for any pair of colors “A” and “B”, the areas colored A and those colored B must be geometrically equivalent, and that there is a symmetry that moves all of A to B. (Possibly ignoring “background” elements which stay fixed.) Furthermore, this remains true if we limit ourselves to using only the “color-respecting” symmetries: symmetries that act as permutations on the colors of the pattern. Thus our traditional idea of a three-color symmetric pattern, with colors A, B, and C, might have color-respecting symmetries that: (i) maps all areas of color C to C, while interchanging colors A and B; or (ii) move all A to B, all B to C, and all C to A; etc. What is *not* allowed, though, is a “color-mixing” symmetry that moves some of A to B and some of A to C. For example, consider the Zulu bead pattern of figure 1. If we were only interested in the circles (beads) of this pattern, then this design is a *pma2* design (translations, vertical reflections, glide reflection, and rotations). However, both the vertical reflections and the glide reflection act to move some of the black beads to white beads, while other black beads are moved to black beads. Thus the only “color-respecting symmetries” or, to be concise, “color symmetries” are the translation and the rotations in the center of each block of 8 beads. Thus the “color symmetry group” of this design is *p112*. Fortunately, that still leaves enough symmetries to interchange the two colors, so this qualifies as a “colored pattern”. It is not, however, a “perfectly colored pattern” – a term which requires *all* geometric symmetries to also be color symmetries. This paper does not require our patterns to be perfectly colored.

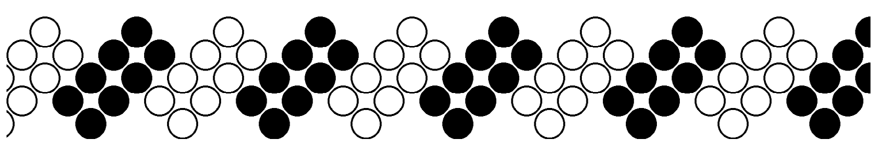


Figure 1: *A Zulu beadwork pattern with pma2 geometric symmetry, but p112 color symmetry.*

[1] *The author wishes to thank Donald Crowe for suggesting this topic, and to the staff of the Logan Museum, at Beloit College, for their aid in this research – both in locating candidates for this analysis and taking the photos for the web version of this work (at https://math.beloit.edu/chavey/).*

A traditionally three-colored pattern must then have three congruent colored portions, each color being symmetrically equivalent to the others. But the more general idea that colorings be done "in coordination with" the symmetries allows another option. Consider the Pashtun (Afghanistan) dress flower of figure 2. Here 50% of the design is red, 25% is blue, and 25% is yellow. A rotation of $1/4^{th}$ fixes the red motifs (at the 4 main compass points) and interchanges the yellow and blue motifs (at the 4 corner points). A rotation of $1/8^{th}$ interchanges reds with the union of yellow and blues. While this is not a traditional three-coloring, it seems to model the symmetric intent of its artist. Furthermore, imagine this beaded flower as seen by someone with blue-yellow color blindness – the 2^{nd} most common form of color blindness. To them, this would be a perfectly 2-colored pattern. Imagine someone whose eyes were more color sensitive than the author's, who might decide that the right and left red motif had a slightly different shade of red than the top and bottom motif. That person would see a perfectly 4-colored pattern. This helps justify our belief that this pattern has been colored "in coordination with" the design symmetries. Schauermann ([3], pp. 134-136) refers to such linear designs as "recurration," although he does not require such designs to be symmetric.

Figure 2: *A 2:1:1 central design of the Pashtun.*

2. Classification of Central and Strip 2:1:1 Patterns

To simplify our notation, we refer to the three colors as A, B.1, and B.2, where A is the color of half the colored portion, while B.1 and B.2 combine to cover the other half. As indicated by our example above, if we view B.1 and B.2 as a single color "B", then the A-B pattern must be a proper two-color pattern. If we look at the subgroup of the color symmetries that fix A, as if we viewed A as part of the background, then the B.1-B.2 pattern must also be a proper two-color pattern. So, to classify such designs, we need to investigate the ways to combine pairings of two-colored symmetry patterns to achieve this type of design. The traditional convention (e.g. [1]) for two-color designs uses names such as *p'112* for figure 1, where the group of all color symmetries is *p112*, the translation *p* reverses colors, and at least some rotations preserve color. This notation is useful for its conciseness and to emphasize color-reversing symmetries. An alternate notation gives the underlying group of color symmetries followed by the group of color symmetries that fix all colors. In this notation *p'112* would be written as *p112 / p112* (where a longer translation preserves colors). These symbols also differentiate the 17 two-color strip patterns, and tend to emphasize the symmetries that fix colors. The two-color strip patterns in this notation are listed in table 1.

p111 / p111	*p1m1 / p1m1*	*p1a1 / p111*	*pma2 / pm11*	*pmm2 / pmm2*	*pmm2 / p1m1*
pm11 / pm11	*p1m1 / p1a1*	*p112 / p112*	*pma2 / p1a1*	*pmm2 / pma2*	*pmm2 / p112*
pm11 / p111	*p1m1 / p111*	*p112 / p111*	*pma2 / p112*	*pmm2 / pm11*	

Table 1: *The 17 two-color strip patterns using the "color group / color fixing group" notation.*

Using this notation for the A–B design, the second group gives us the symmetries which fix A, which is then the name of the *first* group for the B.1–B.2 color design, i.e. symmetries which fix the background, and are color respecting for the B.1–B.2 design. Thus we specify these designs with the notation X / Y / Z, where X is the group of all color symmetries, Y is the symmetry subgroup that fixes A, and Z is the symmetry subgroup that fixes each of the colors. We can then generate all possible symbols by starting with the X / Y symbols from table 1, and replacing Y with all possible "Y / Z" combinations from that table. For example, with an A–B pattern of type *pmm2 / p112*, there are two ways to replace *p112* with a two-color pattern, giving the three 2:1:1 design types: *pmm2 / p112 / p112* and *pmm2 / p112 / p111*. Doing this for all of the two-color designs gives us the 33 design possibilities of figure 4.

The pairs in table 1 represent subgroups of index 2 in the full symmetry group, i.e. a subgroup consisting of half the symmetries of the full group. In our X / Y / Z notation, Z has index 2 in Y, and Y has index 2 in X. Applying this principle to finite (central) designs, there are exactly four ways to generate such sets of nested groups: $D_{4n} / D_{2n} / D_n$; $C_{4n} / C_{2n} / C_n$; $D_{2n} / D_n / C_n$; and $D_{2n} / C_{2n} / C_n$. Figure 2 shows the first type, the second can be made from this by adding one-sided ornaments to it; and figure 3 shows constructed examples of the other 2, with n=2 in all cases.

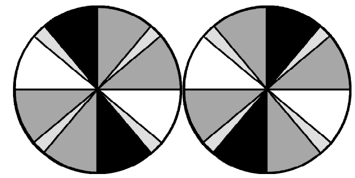
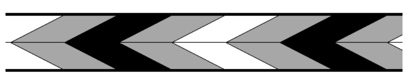


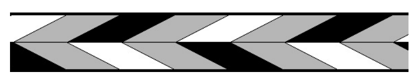
Figure 3: *2:1:1 designs $D_4 / D_2 / C_2$ (left) and $D_4 / C_4 / C_2$ (right).*

Figure 4 shows drawings of the 33 possible 2:1:1 frieze patterns, followed by the number of examples found in our survey. The web shows photos of these 40 examples (21 from the Logan collection and 19 from Jones [2]). An analysis of the large collection of color patterns in Jones found 30 examples of 2:1:1 designs: the 19 frieze patterns, one central design and 10 wallpaper patterns. This tells us that these designs are uncommon, but not rare. However, the distribution of these patterns across cultures is surprising. In Jones, 9 of the 18 frieze patterns, and 7 of the 10 wallpaper patterns were Egyptian designs, with the rest disbursed widely among the 17 other cultures represented in his collection. This implies a strong correlation between this design type and the culture producing them. Fully half of the Logan museum examples came from American Indian woven baskets. These were not correlated with a specific tribe, but does seem to imply a shared sense of willingness to try new and interesting designs, along with a craft form that is amenable to the construction of such designs. For example, only one 2:1:1 pattern type was found among the Logan's fairly large collection of American Indian beadwork. Beadwork has far fewer constraints on the artisans, hence they were less likely to limit themselves to geometric designs. For woven bags geometric designs are more natural, and 2:1:1 designs are more easily found by experiment. An unusual find in the Logan was a Mixtec bag that contained 3 different 2:1:1 frieze design types (figure 4: #1, 13, and 24), plus a flawed attempt at a fourth. Thus at least one historic weaver appears to have been particularly interested in these 2:1:1 designs! As a final cultural note, Jones had only one 2:1:1 central pattern, while the author has 11 examples of such designs from Pashtun beaded dress flowers; all either $D_8 / D_4 / D_2$ or $C_8 / C_4 / C_2$. The Pashtun also use 2:1:1 frieze designs as borders on some dress flowers, so this is an example of a symmetry form that is particularly representative of this one culture.

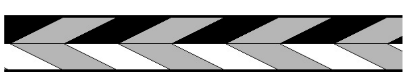
1. *p111 / p111 / p111* (1)



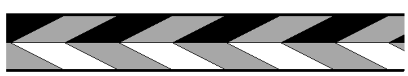
2. *p1m1 / p1m1 / p1m1* (11)



3. *p1m1 / p1m1 / p1a1*



4. *p1m1 / p1m1 / p111*



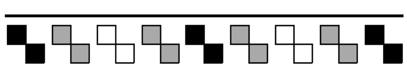
5. *p1m1 / p1a1 / p111*

6. *p1m1 / p111/ p111* (1)

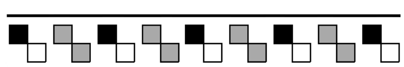
7. *pm11 / pm11 / pm11* (1)

8. *pm11 / pm11 / p111*

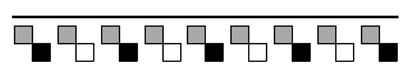
9. *pm11 / p111/ p111*



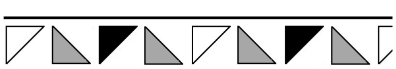
10. *p112 / p112 / p112* (3)



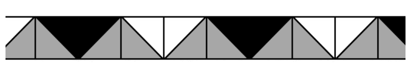
11. *p112 / p112 / p111* (3)



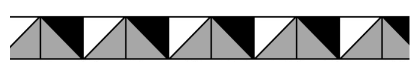
12. *p112/p111/p111* (2)



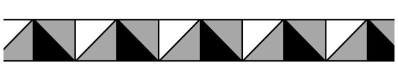
13. *p1a1 / p111 / p111*



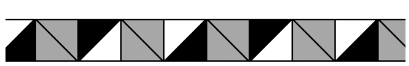
14. *pma2 / pm11 / pm11* (5)



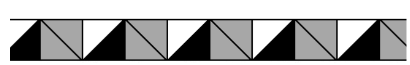
15. *pma2 / pm11 / p111*



16. *pma2 / p1a1 / p111* (1)



17. *pma2 / p112 / p112*



18. *pma2 / p112 / p111*

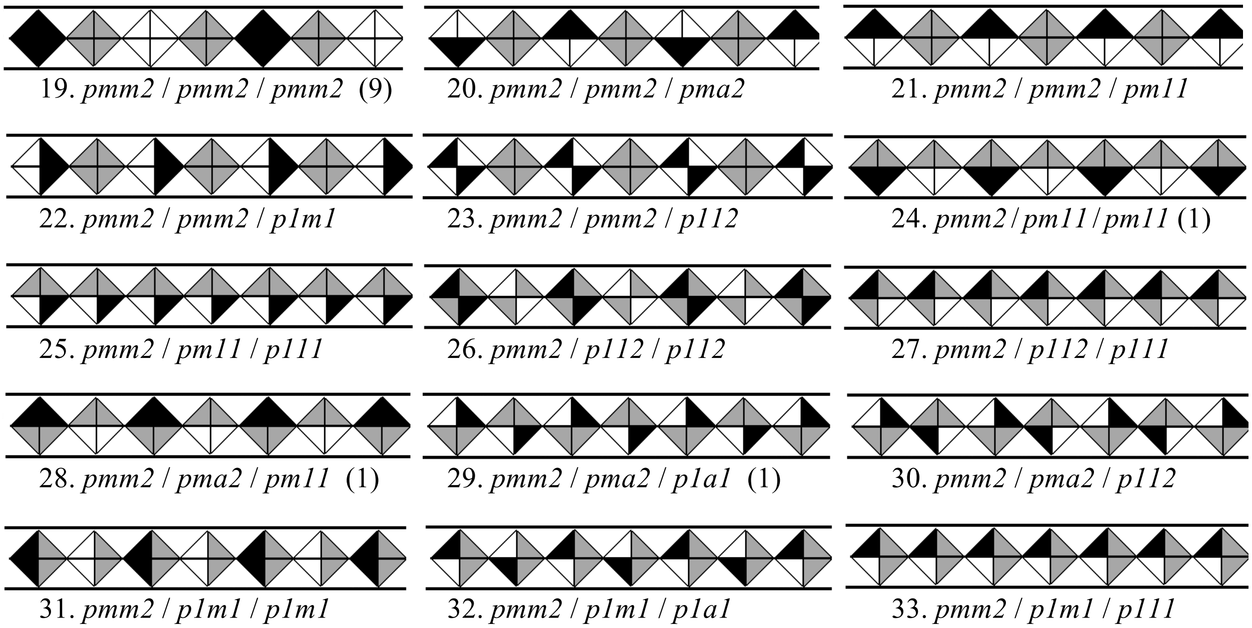


Figure 4: *Constructed examples of the thirty-three 2:1:1 three-colored strip patterns, and the numbers of cultural examples of each found. See http://math.beloit.edu/chavey/ for photographs of examples.*

3. Generalizations of 2:1:1 Patterns

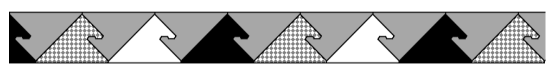


Figure 5: *Navajo-style rug, with 3:1:1:1 color symmetry, in the author's study.*

It is possible to extend these results by looking at other sequences of nested sub-groups with small indexes. For example, figure 5 shows a 4-color pattern from a rug of the author's. Using the notation above, this would be a 3:1:1:1 color pattern of type *p112* / *p111* / *p111*, i.e. with a rotation that interchanges the gray with the other 3 colors. A Mexican shirt in the Logan Museum had a 3:1:1:1 pattern of type *pmm2* / *pmm2* / *pmm2*. A Berber dress in the Logan Museum had a "nearly" 4:2:1:1 four color design like figure 4, #2, but with wedges ordered A–B–A–C_1–A–B–A–C_2–A–B–A–C_1 etc. Unfortunately, the "A" wedge was a bit wider than the others, hence isn't quite symmetric with them. These *k*-colored patterns with $k > 3$ are fairly rare though, and hence may not be worth classifying, although the techniques shown here should work. There are a modest number of examples of 2:1:1 symmetric colored wallpaper patterns, such as the 10 designs in Jones [2], and fabric patterns gingham (or "buffalo check") and many common argyles. A classification of 2:1:1 wallpaper patterns might find only a few cultural exemplars, but might construct some new interesting and attractive patterns.

References

[1] Crowe, Donald and Dorothy Washburn, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, Univ. of Washington Press, Seattle, Washington, 1988.

[2] Jones, Owen, *Grammar of Ornament*, Bernard Quaritch, London, 1910.

[3] Schauermann, F., *Theory and Analysis of Ornament*, Sampson Low, Marston & Co, London, 1892.