Toss and Spin Juggling State Graphs

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Abstract

We review the state approach to toss juggling and extend the approach to spin juggling, a new concept. We give connections to current research on random juggling and describe a professional-level juggling performance that further demonstrates the state graphs and their research.

Introduction

A well-studied mathematical model for juggling called *siteswap* has been described in previous Bridges papers by Bracken [1] and Naylor [14]. The model is essentially one-handed and it only keeps track of the throwing schedules of the objects thrown one by one. While the model can be generalized to include several hands and simultaneous throws, it lacks any spatial information.

Juggling is commonly viewed as *tossing* various objects: Balls, rings, clubs, and their variants such as torches and chainsaws. The siteswap model is purely temporal and regards all these objects as *particles*, and accordingly the mathematical treatment is mostly combinatorial in nature. In practice spatial information is often included separately, for example: "Siteswap pattern 423 with clubs; 4s as double spins, 2s as swings, and 3s as single spins thrown behind the back."

However, there are other common forms of juggling that are more spatial in nature and deserve a mathematical treatment of their own. Although the word spin can be used as a club rotation count like above, we introduce *spin juggling* to include poi spinning, devilstick manipulation, and diabolo juggling. Tossing may occur in these forms of juggling as well, but it is less significant compared to the spinning that consists of various spatial configurations and rotational transitions between them.

In this paper we review the state approach to toss juggling, and introduce a *poi state graph* as an example of a *spin layer* that can be added to toss juggling. We describe the mathematical research related to random walks on juggling state graphs and, finally, depict a juggling performance built on the material in this paper.

Historical background

The mathematical modeling of juggling patterns started around 1980, when computers gained popularity and scholars envisioned robots and simulators. Shannon [16] is probably the first mathematical manuscript on juggling, with emphasis on robotics.

The siteswap model was independently invented by several people in the early 1980s and was first published by Magnusson and Tiemann [12]. Since then a dozen or so serious papers, in addition to a monograph by Polster [15], have been published on the subject. We mention the following papers:

- Graham et al. [2, 3, 4, 5] on the combinatorics of periodic siteswaps
- Ehrenborg and Readdy [7] pointing out a connection between n-periodic siteswaps and the affine Weyl group $\widetilde{A}_{n-1}$ (along with further interesting combinatorics)
- Warrington [17] on uniformly random siteswaps with bounded throws
- Devadoss and Mugno [6] on generating braids via siteswaps
- Leskelä and Varpanen [11] on general random siteswaps
- Knutson, Lam and Speyer [10] decomposing the Grassmannian manifold of all k-planes in n-space via n-periodic siteswaps with k particles.

There are also various toss juggling simulators such as Juggling Lab [9] coordinated by Jack Boyce.

Siteswap state graphs

Siteswap juggling patterns are usually modeled as bijections $f\colon \mathbb{Z} \to \mathbb{Z}$ with $f(t) \geq t$ for each $t \in \mathbb{Z}$, the interpretation being that a juggling pattern is eternal, a particle thrown at beat t is scheduled to be next thrown at a future beat $f(t)$, and that at most one particle is allowed to be thrown at a given beat. Here the number $f(t) - t \geq 0$ is called the *throw* at beat t, and a zero-throw is interpreted as an empty throw where the juggler has nothing to throw and waits. A nontrivial orbit of f corresponds to the throw times of a particle, and the long-time average of the throws equals the number of particles. The model is essentially one-handed, but it also comprises the commonly encountered patterns with alternating hands.

Figure 1 depicts a three-particle, period-three pattern. Instead of beat numbers t we have listed the *throw values* $f(t) - t$; the pattern is accordingly called 450.

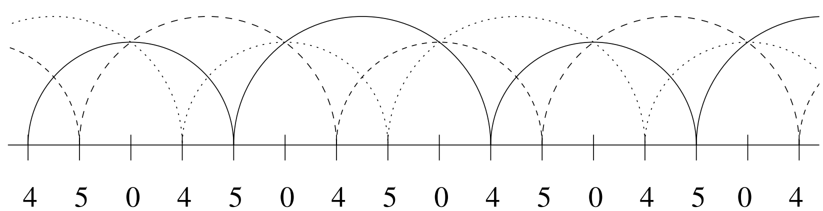

Figure 1 : *The pattern* 450*.*

Just before a 5-throw is performed in Figure 1, the three particles in the air are scheduled to be next thrown zero, two and three beats from "now", the vertical line in Figure 2. We say that the juggler is at *state* $\{0, 2, 3\}$.

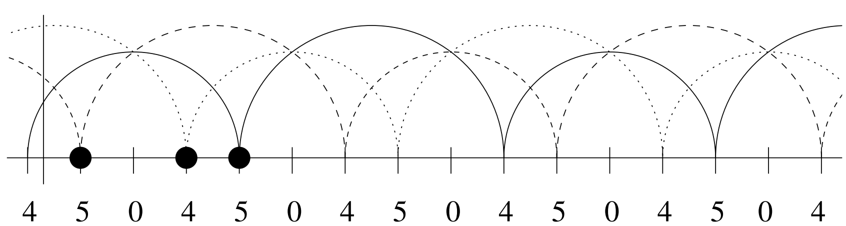

Figure 2 : *A state* $\{0, 2, 3\}$ *in the pattern* 450*.*

As the 5-throw is performed and time advances one beat forward, the state changes from $\{0, 2, 3\}$ to $\{1, 2, 4\}$ (Figure 3):

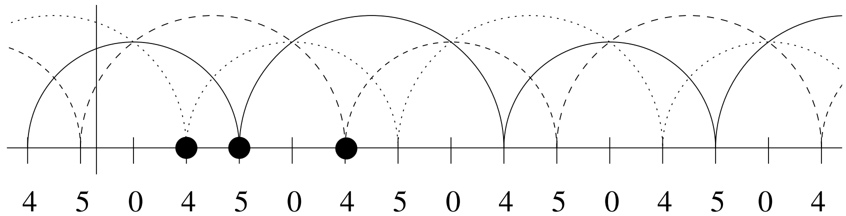

Figure 3 : *A state* $\{1, 2, 4\}$ *in the pattern* 450*.*

The pattern 450 can hence be represented via its *state cycle* as in Figure 4 below. The states have been drawn vertically to reflect particles falling down and thrown up. The states $\{0, 1, 3\}$, $\{0, 2, 3\}$ and $\{1, 2, 4\}$ are three-element subsets of the five-element set $\{0, 1, 2, 3, 4\}$, because there are three particles and the maximum throw value is five. The throw values 4, 5 and 0 yield transitions from one state to the next.

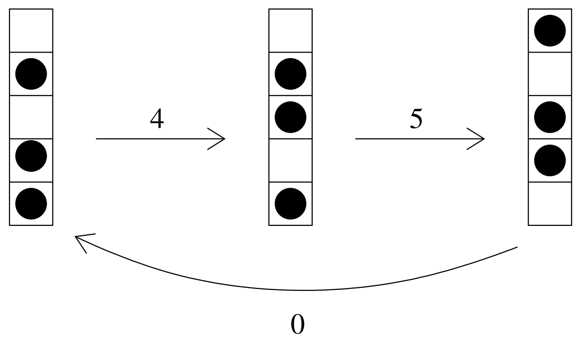

Figure 4 : *The state cycle for the pattern* 450*.*

Now any three-particle pattern with maximum throw five is a path in the following *state graph* (Figure 5):

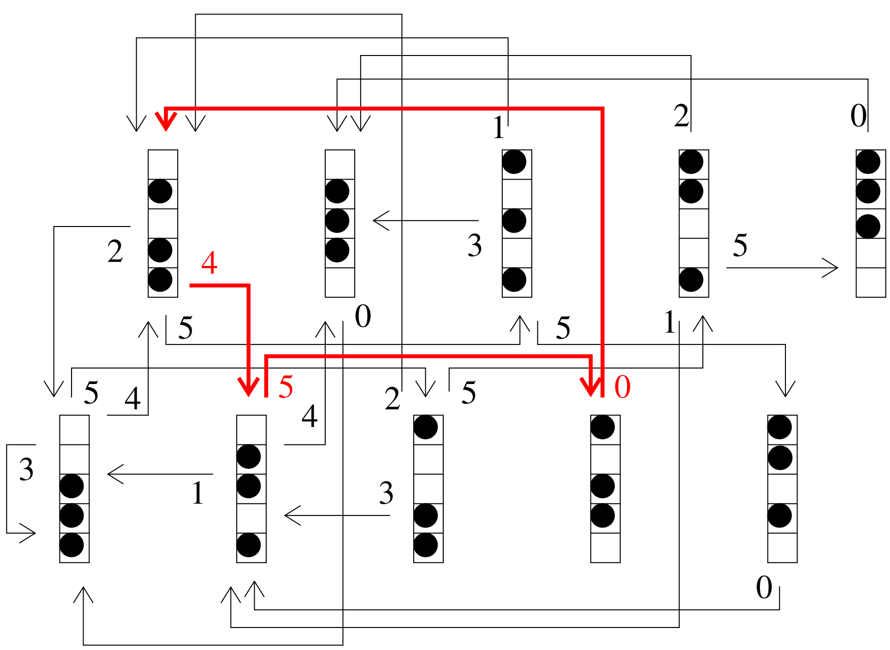

Figure 5 : *The state graph for three particles and maximum throw five. The pattern* 450 *is highlighted.*

The state model was independently invented by Jack Boyce and Allen Knutson in 1988 and is widely used by jugglers to communicate juggling patterns and transitions between them. The model naturally generalizes to allow simultaneous throws (*multiplex juggling*) and many hands (*multihand juggling*) as explained in Polster [15].

Spin juggling

The simplest form of spin juggling is *poi spinning*, where a juggler spins tethered weights, one in each hand, through a variety of rhythmical and geometric patterns. Figure 6 shows a schematic poi juggler in its simplest state: The tethers are not braided. The weights rotate 180 degrees per beat in paths that resemble half-circles: One weight from down to up, the other from up to down. Braiding of the tethers occurs when one of the weights moves from one side of the juggler's body to the other.

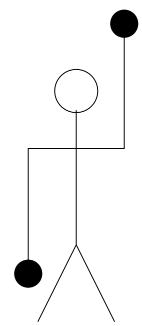

Figure 6 : *A poi juggler.*

In order to capture the essence of poi spinning, we ignore the weights and regard the tethers as extended hands (made of a rubber string, say). We also ignore the juggler's body except from the fixed shoulder line. We assume that the juggler is viewed from behind and that the direction of the spinning is forward: The rotation of the weight from up to down takes place in front of the juggler's body, rather than behind. We also make the simplifying assuption that a hand may move to the other side of the body (shoulder line) only while rotating from down to up.

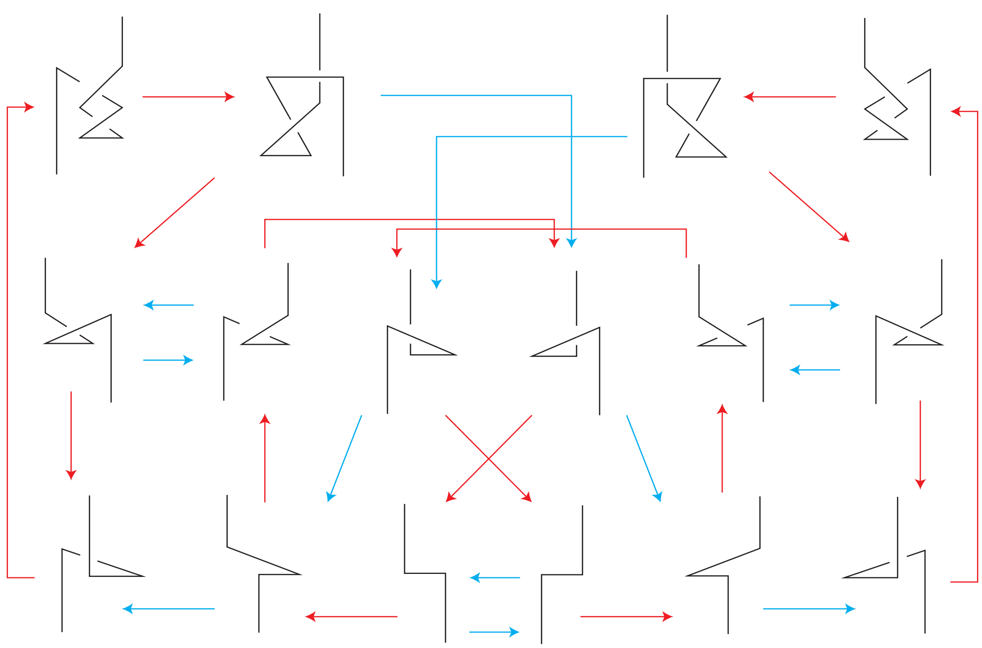

Figure 7 : *A poi state graph.*

The resulting poi state graph is given in Figure 7, where each state (black line) consists of a fixed horizontal shoulder line along with a changing hand configuration of the juggler. The arrows indicate the 180-degree rotations of the hands from down to up and vice versa. A red arrow indicates that a hand moves to the other side of the body, while a blue arrow indicates that no such move takes place. (Remember that only the hand rotating from down to up is allowed to move to the other side of the body.) The braiding of the hands is bounded to a maximum of two crossings, enough to cover the commonly encountered ”3-weave” and ”5-weave” poi spinning patterns.

For example, if a move is denoted by R (red) and a non-move by B (blue), then the ground pattern formed by the two states in the middle bottom row of Figure 7 is . . .BBB. . . or, assuming that the symbols repeat periodically, simply B. The 3-weave pattern is similarly RRB and the 5-weave pattern is BRRBB, but these patterns can not be started directly from the ground pattern. Starting from the ground pattern and including the default transition moves in parentheses, the 3- and 5-weave patterns read (R)RRB(RRR) and (R)BRRBB(BRRBR), respectively.

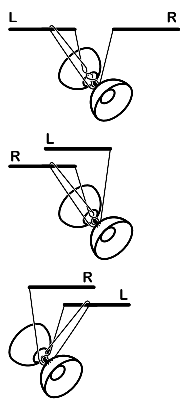

Figure 8 : *A few diabolo states.*

The poi hand movements form a spatial layer that can be combined with siteswap. For example, the siteswap 522 combined with the 3-weave RRB is a nice pattern when juggled with clubs. In addition, the poi hand movements are similar to the hand movements that occur in devilstick juggling or staff manipulation (these are considered mathematically equivalent), so the poi state graph applies to these forms of juggling as well. Finally, similar state graphs have been considered for the diabolo by Männistö [13]. Figure 8 shows some states for the diabolo, the complete state graph being too complicated to be explained in this paper.

Random juggling

Although no one is able to perform truly random juggling, one may still consider random juggling from a mathematical point of view. Random walks on juggling state graphs are fascinating because a juggling state graph possesses a clear mathematical structure: Nontrivial, but not too complicated to produce exact results.

Given a juggling state, consider all its admissible follower states. For example, the leftmost state in Figure 5 has three admissible followers (reached by transitions labeled 3, 4, and 5), one of which is itself. Associate to each follower state a probability such that the probabilities sum to one. In other words, choose a transition from the current state to the next state by throwing a dice. Assume that the dice is known at each state; then ask: ”*How popular, relative to each other, are the juggling states in the long run?*” In other words, one asks for the *steady-state distribution* (or *equilibrium distribution*) of the random juggling process.

While it is often possible to calculate numerical approximations of equilibrium distributions, explicit formulas are typically considered difficult or impossible. However, the structure present in juggling state graphs allows for some explicit formulas, the first of which was obtained by Warrington [17] in the case where each siteswap throw is chosen uniformly randomly: If there are, say, three alternatives for the throw whenever the juggler does not wait (as is the case in Figure 5), then each alternative is chosen with probability one third. The exact long-term visit frequencies are shown in Figure 9 and calculated using the formula in Theorem 1.

Theorem 1 ([17] Thm. 5)**.** *In uniform random juggling with k particles and maximum throw m, the frequency of a state $B \subset \{0, 1, \ldots, m-1\}$, $|B| = k$ is*

$$\left\{ {m+1 \atop m+1-k} \right\}^{-1} \prod_{x \in B} |\, \{x, \ldots, m\} \setminus B \,|,$$

where $\left\{ {m+1 \atop m+1-k} \right\}$ is a Stirling number of the second kind.

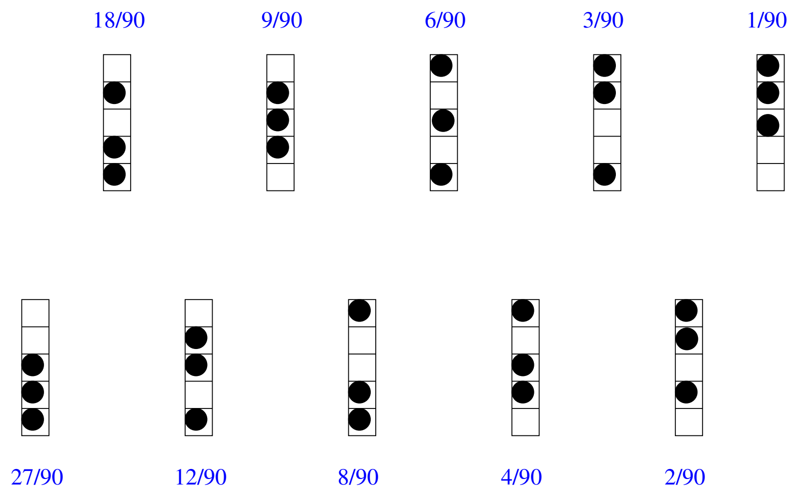

Figure 9 : *The visit frequencies of the states in uniformly random siteswap juggling with three particles and maximum throw five. The (leftmost) state with lowest potential energy is visited almost every third beat whereas the (rightmost) state with highest potential energy is visited only once in every 90 beats.*

In our recent research [11, 8] we have considered situations where the dice is not fair and where the throws can be arbitrary high. Explicit formulas are available in these situations as well, and new kinds of combinatorial limit formulas emerge when the maximum throw approaches infinity. Our ongoing research involves juggling with antimatter and mean field juggling involving spinning hands. These are further demonstrated in my presentation.

The presentation

I started studying mathematics in 1994 and found juggling in 1998. Since then my main interest has been in the mathematical models of both toss and spin juggling. I have kept practicing regularly, and juggle fluently (but not simultaneously) with the six most essential toss and spin equipment: balls, rings, clubs, poi, devilstick, and diabolo.

I have built a half-hour mathematical juggling performance around the topics surveyed in this paper. The performance has the form of a lecture, but the speech is prerecorded in the slides that I change with a foot pedal while quietly demonstrating various juggling patterns; see the screenshot in Figure 10. A three-minute trailer of the performance in Figure 10 is available (as of May 4, 2014) at

`http://www.youtube.com/watch?v=u2mErXtXqMc`

(or search ”Varpanen Heureka 2013”). The trailer is in Finnish with English subtitles, but the performance is available in English as well.

My aim is to illustrate for a general scientific audience in an artistic, entertaining and graspable way how hands-on juggling models relate to advanced mathematical research. During the last ten years I have constantly improved the show and performed it on various occasions such as college mathematics camps, science slam events, mathematics teachers' conferences, science center mathematics days, and university mathematics course lectures. The performance can be carried out in any normal auditorium or lecture hall with a high enough ceiling and some free floor space.

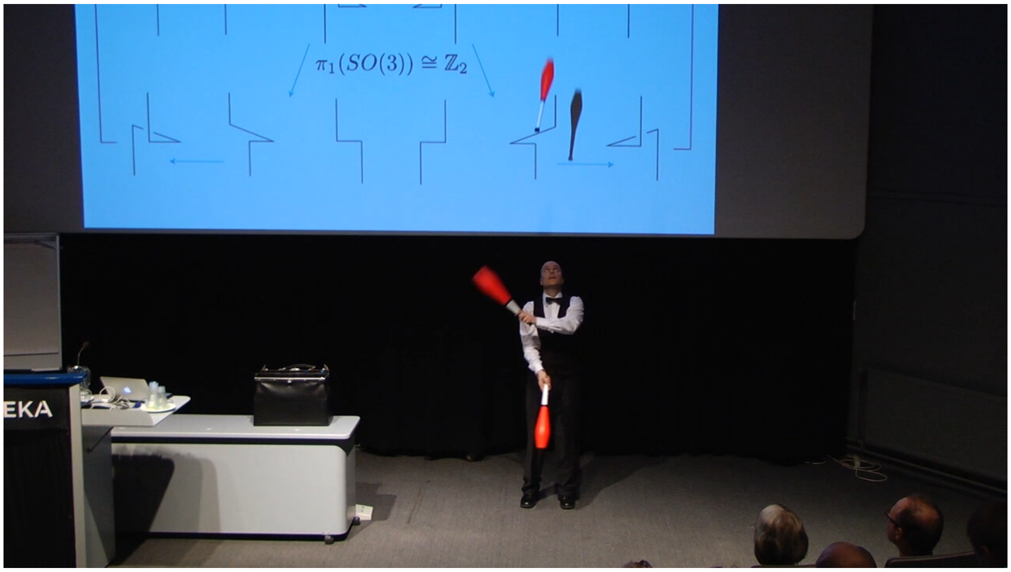

Figure 10 : *A screenshot of my juggling lecture from 2013. The white foot pedal can be seen on the floor at the desk.*

Conclusion

State graphs (also called state diagrams) occur everywhere: Population dynamics, the Internet, quantum physics, factories, etc. Juggling state graphs are free from industrial restrictions and give rise to rich, deep and beautiful mathematics. In addition, juggling provides an exceptional means of communicating the mathematics to a wider audience.

While toss juggling state graphs have been analyzed for some decades, spin juggling is a new concept that adds spatial and rotational dimensions to juggling. Currently its mathematical formulation is still in its infancy, and no serious spin juggling simulators exist (to the best of my knowledge). Besides mathematical papers on spin juggling, I would love to witness a scientific juggling simulator that generates and analyzes random walks on both toss and spin juggling state graphs, while showing the patterns on the screen.

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References

[1] C. Bracken. Counting The Number of Site Swap Juggling Patterns with Respect to Particular Ceilings. *Proceedings of Bridges 2008: Mathematics, Music, Art, Architecture, Culture*, 219–224, 2008.

[2] J. Buhler, D. Eisenbud, R. Graham and C. Wright. Juggling drops and descents. *Amer. Math. Monthly* 101 (6), 507–519, 1994.

[3] S. Butler and R. Graham. Enumerating (multiplex) juggling sequences. *Ann. Comb.* 13 (4), 413–424, 2010.

[4] F. Chung, A. Claesson, M. Dukes and R. Graham. Descent polynomials for permutations with bounded drop size. *European J. Combin.* 31 (7), 1853–1867, 2010.

[5] F. Chung and R. Graham. Primitive juggling sequences. *Amer. Math. Monthly* 115 (3), 185–194, 2008.

[6] S. Devadoss and J. Mugno. Juggling braids and links. *Math. Intelligencer* 29 (3), 15–22, 2007.

[7] R. Ehrenborg and M. Readdy. Juggling and applications to q-analogues. *Discrete Math.* 157, 107–125, 1996.

[8] A. Engström, L. Leskelä and H. Varpanen. Geometric juggling with q-analogues. Submitted. Preprint available at *arXiv:1310.2725.*

[9] The Juggling Lab simulator. Online at `http://jugglinglab.sourceforge.net` (as of May 4, 2014).

[10] A. Knutson, T. Lam and D. E Speyer. Positroid varieties: Juggling and geometry. *Compos. Math.* 149 (10), 1710–1752, 2013.

[11] L. Leskelä and H. Varpanen. Juggler's exclusion process. *J. Appl. Probab.* 49 (1), 266–279, 2012.

[12] B. Magnusson and B. Tiemann. The physics of juggling. *Phys. Teach.* 27, 584–588, 1989.

[13] S. Männistö. *A graphical juggling notation system.* Master's Thesis (in Finnish), University of the Arts Helsinki, 2012.

[14] M. Naylor. The mathematical art of juggling: using mathematics to predict, describe and create. *Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture*, 33–40, 2012.

[15] B. Polster. *The Mathematics of Juggling.* Springer Verlag, 2003.

[16] C. Shannon. Scientific aspects of juggling. Manuscript from ca. 1980. Published in *Claude Elwood Shannon: Collected Papers*, pp. 850–864, Wiley 1993.

[17] G. S. Warrington. Juggling probabilities. *Amer. Math. Monthly* 112 (2), 105–118, 2005.