Closed Loops with Antiprisms

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Abstract

This paper presents a variety of polyhedral structures consisting of antiprisms that connect together to form closed loops. They suggest interesting architectural possibilities and pose an open problem to mathematicians: to prove that the results, obtained visually with 3D design software, are mathematically exact.

Attaching *p*-antiprisms

With side length fixed, as the number of sides of a regular polygon increases, the area of the polygon becomes larger and the vertex angles become more obtuse (Figure 1). In antiprisms the dihedral angle between a base and an adjacent triangle becomes more obtuse as the number of sides increases. A p-antiprism is a polyhedron consisting of two regular p-gon bases and $2p$ equilateral triangles, arranged so one base and three triangles meet at each vertex. The bases are parallel and if p is odd then opposite triangles are parallel. As a special case, if p is 3, we obtain the (regular) octahedron. As p becomes larger, the dihedral angle between the top or bottom faces and an equilateral triangle becomes closer to a right angle. Consequently, if an initial 5-antiprism is placed on a ground plane and another 5-antiprism is attached face-to-face on a triangular face, the dihedral angle that the second 5-antiprism makes with the base plane will be less than the corresponding dihedral angle in the same construction made with octahedra (and the dihedral angle would be greater than if we had used 7-antiprisms). In general, we call the second antiprism which is attached to the initial antiprism the "index antiprism". In Figure 2 the initial antiprism (pointed by arrow) starts on the ground plane which on its triangular face connected index antiprism is highlighted. We can attach a third antiprism to the index antiprism on the triangular face opposite the face we used to connect the index antiprism to the initial antiprism. We can continue adding antiprisms on opposite triangular faces, forming a "line" of antiprisms coming out of the initial antiprism, as shown in Figure 2. Depending on which triangular face of the initial antiprism is used, this line of antiprisms will either rise above the ground plane or go down into it.

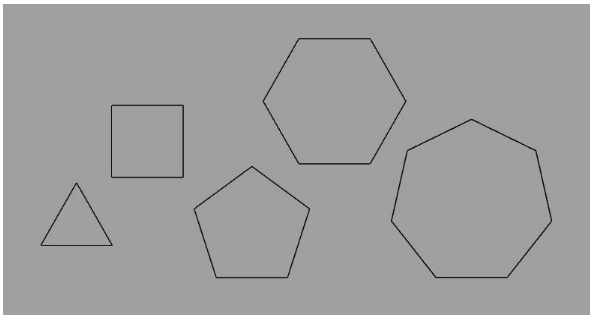

Figure 1: *Regular polygons.*

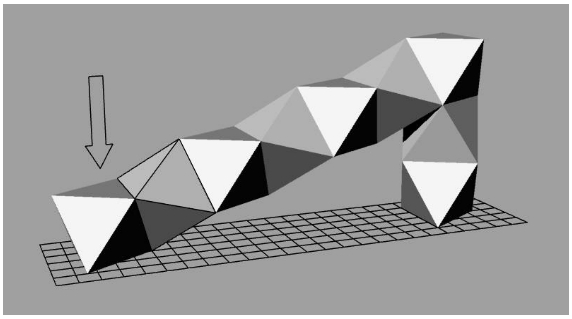

Figure 2: *Line of seven 3-antiprisms*

These coherences give all the *p*-antiprisms, with *p* odd, a uniform property of rising above the ground plane: Figure 2 illustrates that in the line of 3-antiprisms starting with the index antiprism, alternating antiprisms have always parallel faces and rise 2/3 the height of an antiprism with respect to each other (for further explanations consult [2, p. 14, ex. 1]). In this way the seventh antiprism is 6/3 the height of the first above the ground plane, and we can place a stack of two antiprisms between the seventh antiprism and the ground plane. Using the same system as in Figure 2 with 5-antiprisms, we obtain Figure 3: by attaching lines of antiprisms from other triangular faces of the initial antiprism, we can form a 10-sided cycle of ten stacks of antiprisms connected diagonally by lines of 5-antiprisms (clarified by the highlighted sides of antiprisms). If the line starts by sharing an edge with the top of the stack, the line goes upward; if the line only shares a vertex with the top of the stack, the line goes downward. In this way the lines alternately go up and down.

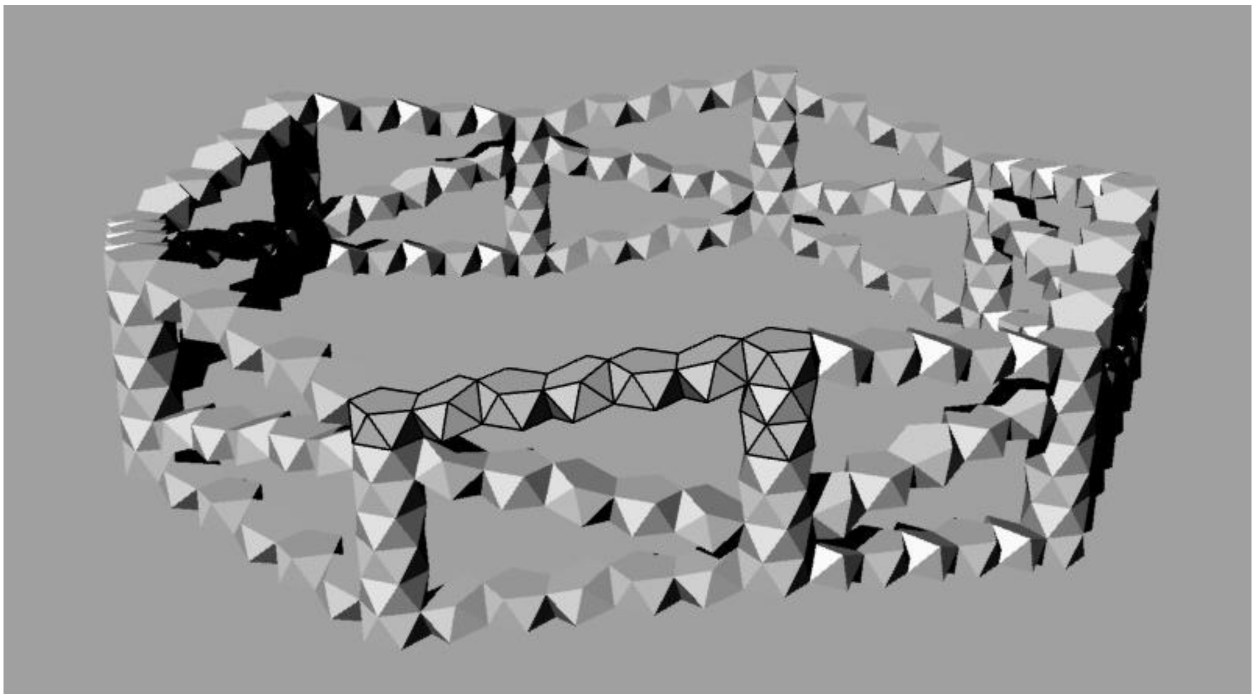

Figure 3: *10-sided cycle constructed with 5-antiprisms*

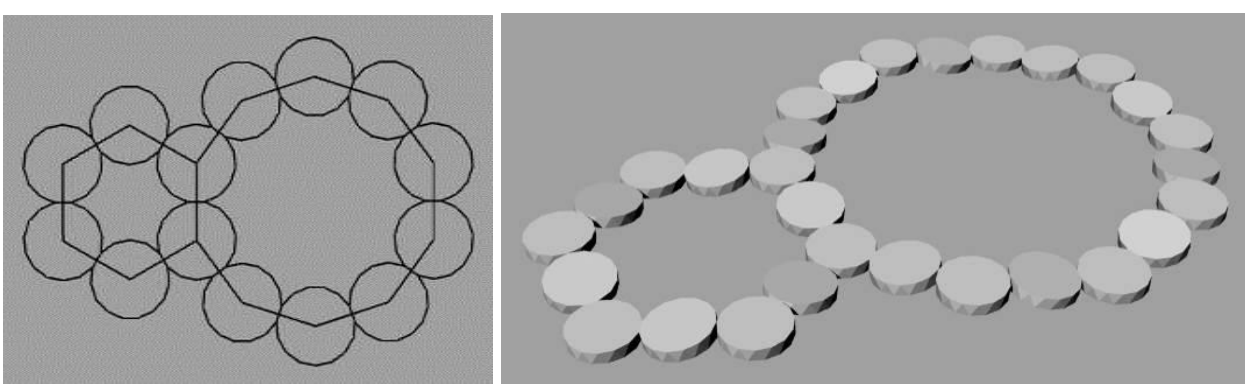

Figure 4: *6-fold and 10-fold cycles of 15-gons (left). Same cycles with 15-antiprisms (right)*

These up and down cycles have a number of sides that depends on *p*, if *p*-antiprisms are used. To clarify with an example: if we set *p* is 5, we obtain the 10-fold construction shown in Figure 3. In general, a cycle built of *p*-antiprisms has 2*p* sides if *p* is prime. If *p* is composite and *d* is a divisor, one can form a cycle with 2*d* sides. Figure 4 (left) shows 6-fold and 10-fold cycles for the 15-gons. Figure 4 (right) shows that for a 15-antiprism we can form 2 cycles: one formed of six 15-antiprisms and another formed of ten 15-antiprims. Furthermore, if *p* is divisible by 3, one can make a triangular "cycle" of seven *p*-antiprisms with three on each edge, and the last one 6/3 height units above the first, so that another antiprism fits in between the first and the last, as shown in Figure 5. This cycle can be repeated, producing an infinite spiral or corkscrew of antiprisms. In fact such corkscrews can share antiprisms, forming an infinite "knot" of alternating clockwise and anti-clockwise oriented corkscrews (with parallel axes). This is also shown in Figure 6 for 9-antiprisms.

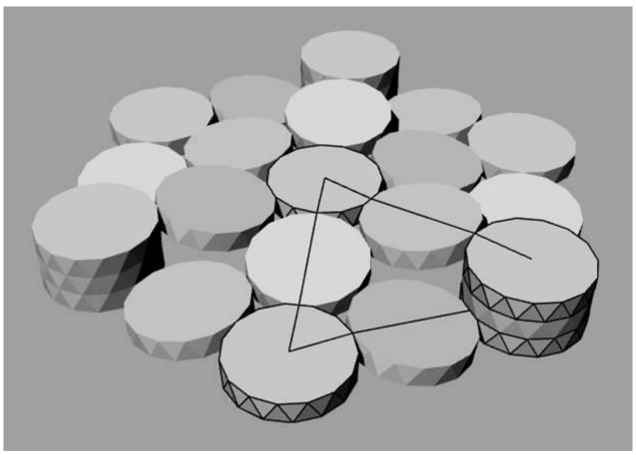

Figure 5: *15-antiprisms infinite knot.*

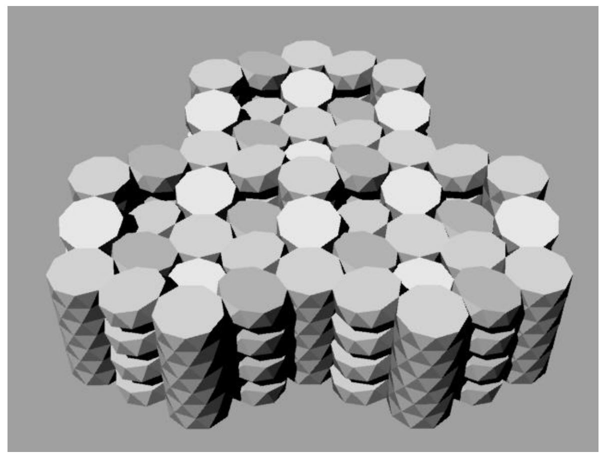

Figure 6: *9-antiprisms infinite knot.*

Making knots by attaching odd and even antiprisms

Another way of making infinite knots is by using both odd and even antiprisms. In this case the odd antiprisms are used as the index antiprisms whereas the even antiprisms remain in the same orientation with their bases parallel to the ground plane. Figure 7 (left) shows 5-gons and 10-gons which become an anti-clockwise corkscrew if they are translated into antiprisms as illustrated in Figure 7 (middle); Figure 7 (right) depicts an extension of this construction. The rule of a 2/3 difference in height, which applies to odd antiprisms of one kind, does not apply in this situation so there is no possibility of fitting an antiprism in between a first and last 10-antiprism in a cycle.

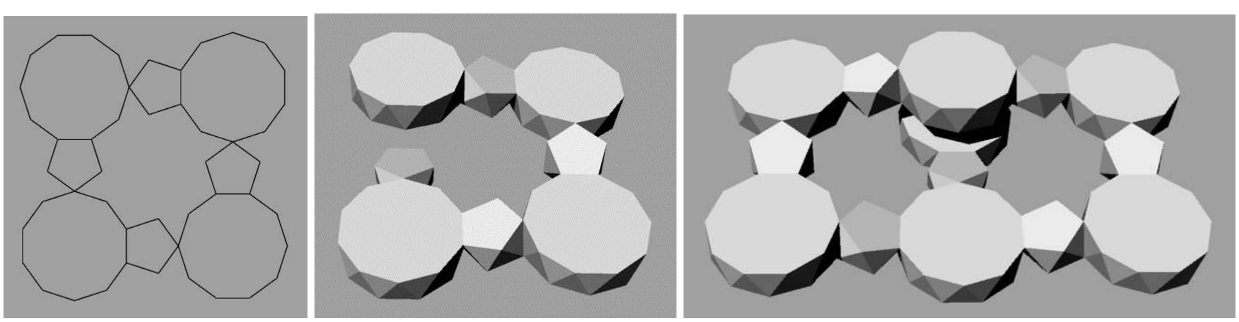

Figure 7: *5-and-10-gons cycle (left). Cycle of 5-and-10-antiprisms (middle). More extended cycle (right).*

Interestingly, the system works with any choice of odd and even antiprisms. For example, Figure 8 (left) shows a pattern of 5-gons and 8-gons, which is translated into a configuration of 5-antiprisms and 8-antiprisms in Figure 8 (right). Even sided *p*-gons with *p* divisible by 4 translate into rhombic cycles of antiprisms due to the positions of their sides and vertices. When viewed from the top, these constructions show alternating (rhombic) structures of clockwise and anti-clockwise corkscrews as in Figure 8 (right).

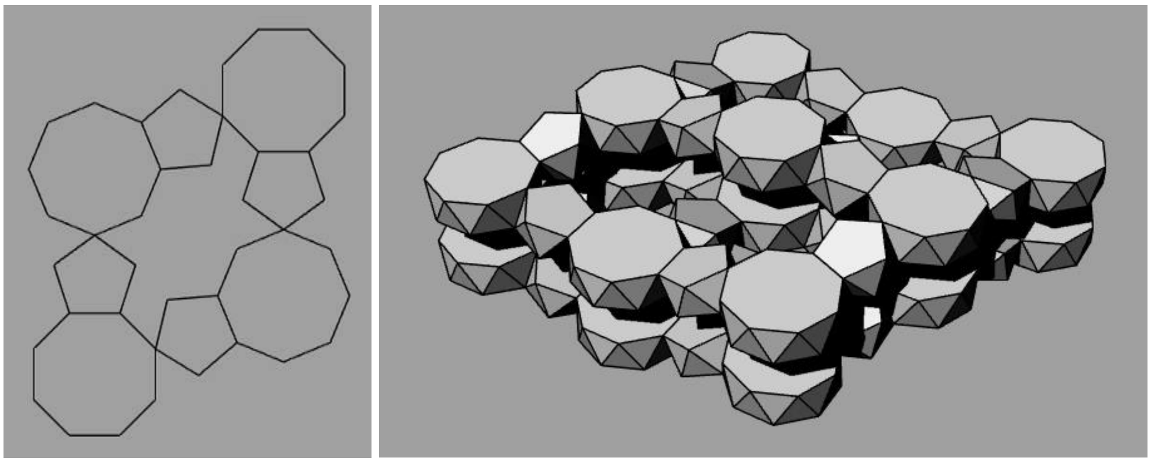

Figure 8: *5-and 8-polygons cycle (left). Rhombic infinite knot with 5-and 8-antiprisms (right).*

Corkscrews of odd antiprisms

By using only one kind of an odd *p*-antiprism, a corkscrew can be made which cycle or cross-section has the shape of a *d*-gon, where *d* divides *p*. A corkscrew becomes a closed structure by stacking a pile of antiprisms on top of the initial antiprism. Since two antiprisms can be stacked underneath the seventh antiprism, the base of the stack is on the base plane of the initial antiprism with a straight line. The height of such a pile of antiprisms depends on the antiprism which is used to form a corkscrew and on the number of antiprisms per side of a corkscrew. When a corkscrew is composed of *p*-antiprisms where *p* is composite, a pile of antiprisms can be stacked on top of the initial antiprism that reaches the bottom of the last antiprism that completes the cycle. Figure 9 (left) shows this as a series of 9-gons cycles with three (center), five (middle) and seven (outside) 9-gons per side which cycle of three 9-gons per side is translated into a 9-antiprism corkscrew in Figure 9 (right). This is always possible except for *p*-antiprisms with *p* prime. Figure 10 (left and middle) shows two groups of 5-gons with three and five polygons per side. Translated into antiprism-corkscrews, these will not fit with a pile of antiprisms placed right above the initial antiprism since the distance of three or five antiprisms does not result in an integer multiple of the anitprism height. In that case, the pile of stacked antiprisms should be placed beside the corkscrew as in Figure 11. With seven 5-gons as in Figure 10 (right) and 13, 19, 25, … antiprisms per side the pile can be stacked right underneath a first antiprism. This is always the case with a corkscrew with a prime number of sides. Figure 12 shows a configuration with seven 7-antiprisms on the sides of a corkscrew with a 7-gon as a cross-section (the same diagonal lines have been used as in Figure 3).

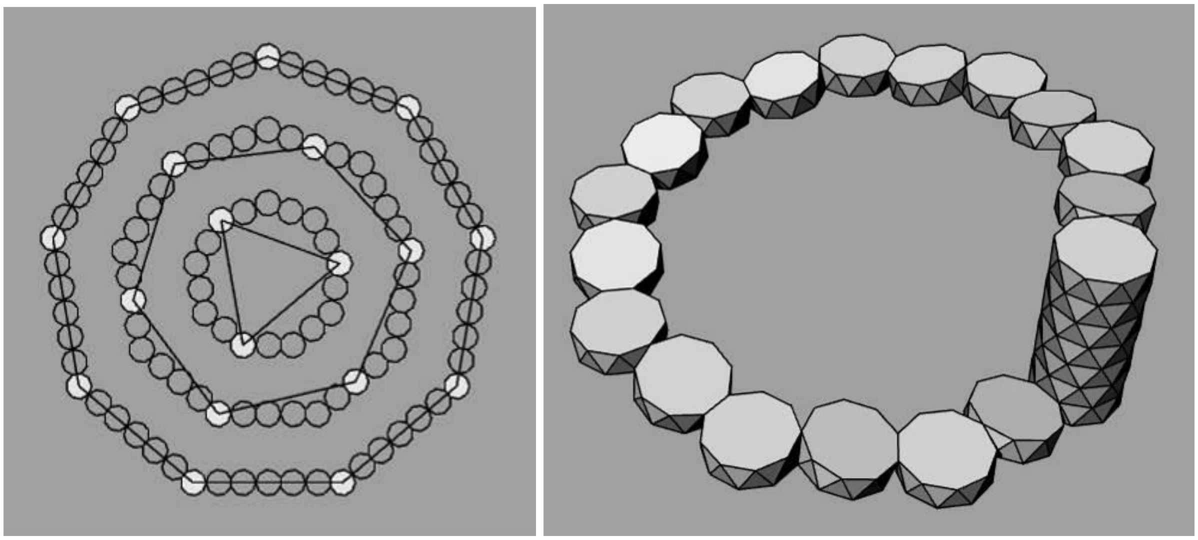

Figure 9: *9-gons cycles (left). 9-antiprism corkscrew; three per side (right).*

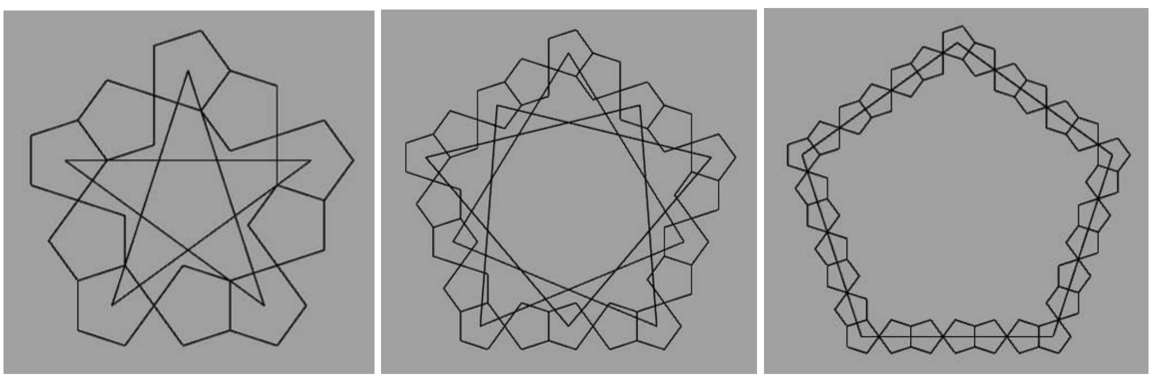

Figure 10: *5-gons cycles; three per side (left), five per side (middle), seven per side (right).*

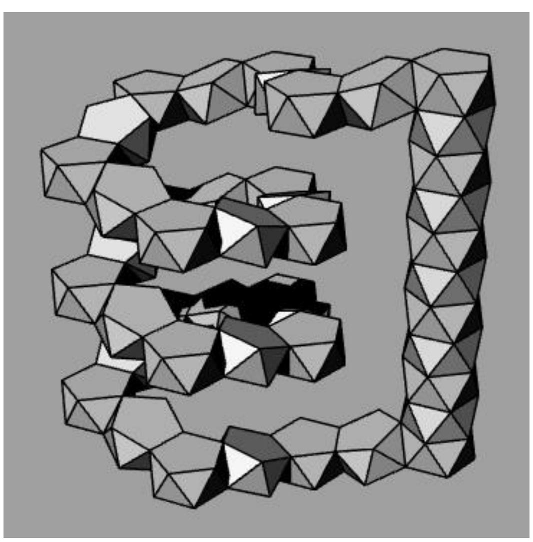

Figure 11: *5-antiprism corkscrew, three per side.*

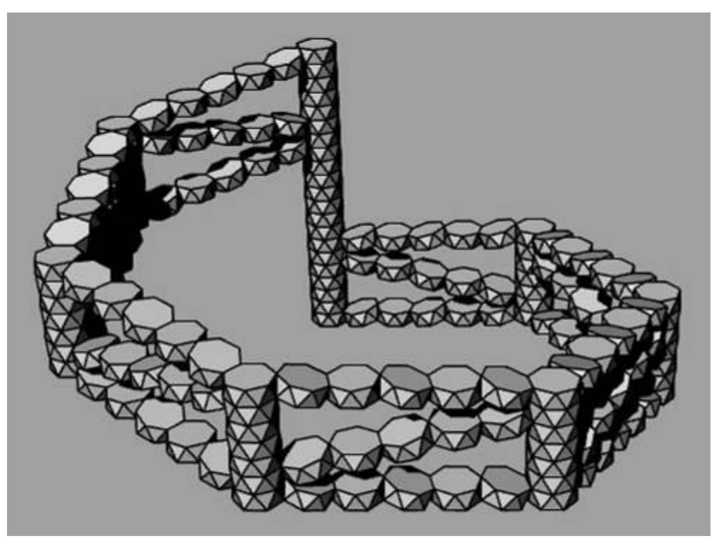

Figure 12: *7-antiprism corkscrew, seven per side.*

Sides of base	Antiprisms per side of corkscrew								
	3	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
3	2	4	6	8	10	12	14	16	18
5	10	20	10	40	50	20	70	80	30
7	14	28	14	56	70	28	98	112	42
9	6	12	18	24	30	36	42	48	54
45	30	60	90	120	150	180	210	240	270

Table 1: The entries are the number of *p*-antiprisms in the stack above the initial antiprism that connect to the last antiprism. The gray entries indicate that the connecting stack will not fit and must be placed to the side.

Constructions with a twist

In the above constructions antiprisms are always connected so an edge of a base of one meets an edge of the other. Since the index antiprisms are attached on their equilateral triangular faces, they can be connected in three ways: in unchanged position or twisted left or right. By turning an index antiprism left or right in a configuration such as in Figure 13 (upper left) more constructions are possible (note that this configuration is comparable to [1, p. 5, F. 8 (left)]. A modification with the index antiprisms twisted right is shown in Figure 13 (bottom left). In this way different 3D rhombic cubes of antiprisms can be obtained as shown in Figure 13 (right).

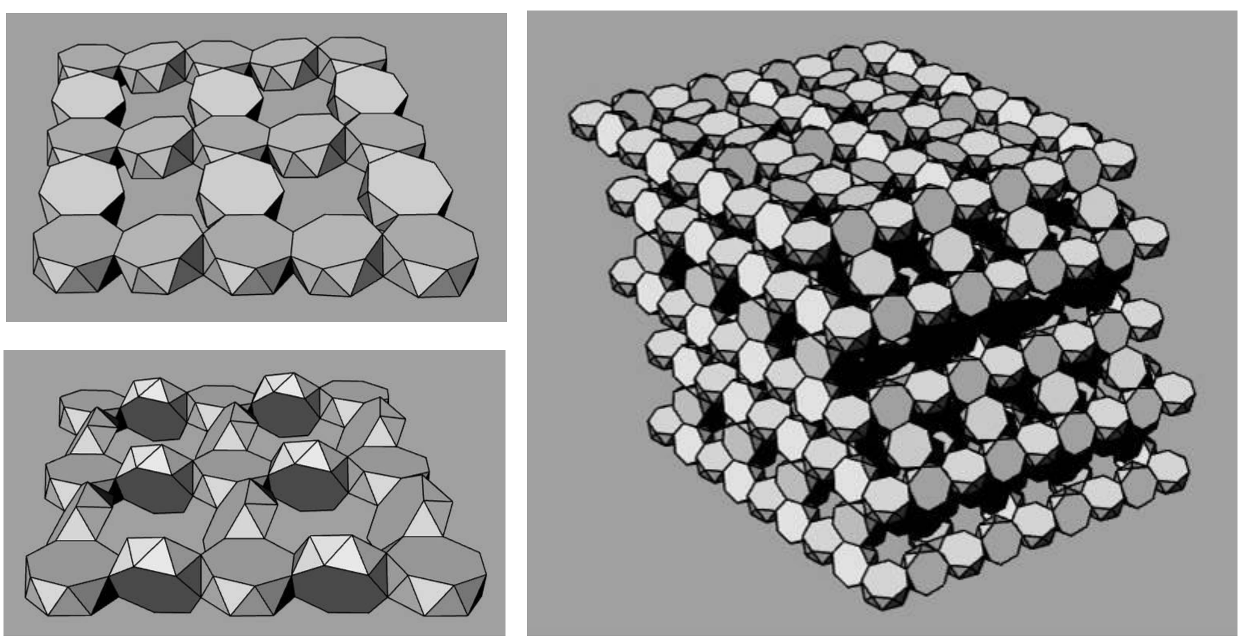

Figure 13: *Platform of 7-antiprisms (upper left). Twisted 'index' antiprisms (bottom left). Five platforms of 7-antiprisms connected by their twisted index antiprisms (right).*

Platonic solids created with antiprisms

If a 3-antiprism (an octahedron) used as an index antiprism, is turned by 120 degrees about the center of its attaching face, there would be no change since the octahedron has 3-fold symmetry about the centers of its faces. However, the octahedron can be used to create corkscrews and infinite knots in the same way that *p*-antiprisms, where 3 divides *p*, can be used to create such structures, as shown in Figures 5 and 6. A pile of 3-antiprisms stacked underneath an initial 3-antiprism has got the same length as a 'line' of 3-antiprisms attached on a triangular face on a side of a 3-antiprism. In this case, we obtain an octahedral sphere as seen in Figure 14, which is in fact a fragment of a 3-antiprism infinite knot. An infinite knot of 3-antiprisms has got the same symmetry as the space filling combination of tetrahedra and octahedra.

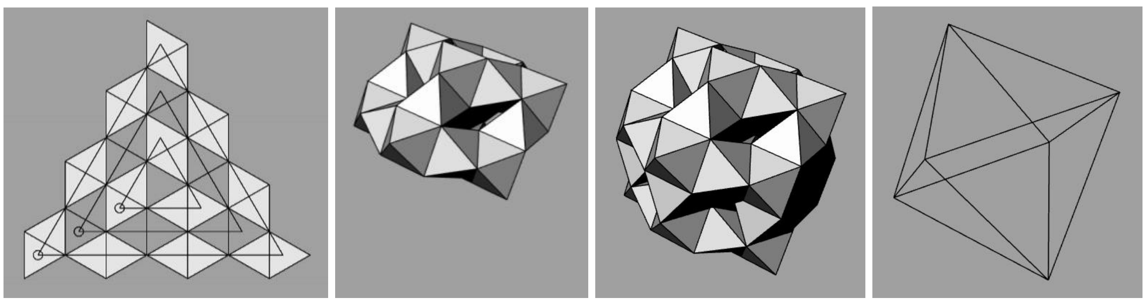

Figure 14: *From left till right; 3-gons cycles, knot***,** *octahedral sphere***,** *octahedron*

By twisting a 5-antiprism left and right on their equilateral triangular faces as shown with 7-antiprisms in Figure 13 it is the only one of all antiprisms which sides exactly touch; six 5-antiprisms fit inside an icosahedron. If *p*-antiprisms are used with *p* greater than 5, a gap will appear. Icosahedral-and-related spheres can thus be constructed with 5-antiprisms as shown in Figures 15, 16 and 17. The antiprisms which connect the individual cycles are intersected in Figures 15 and 16 with an icosahedral angle.

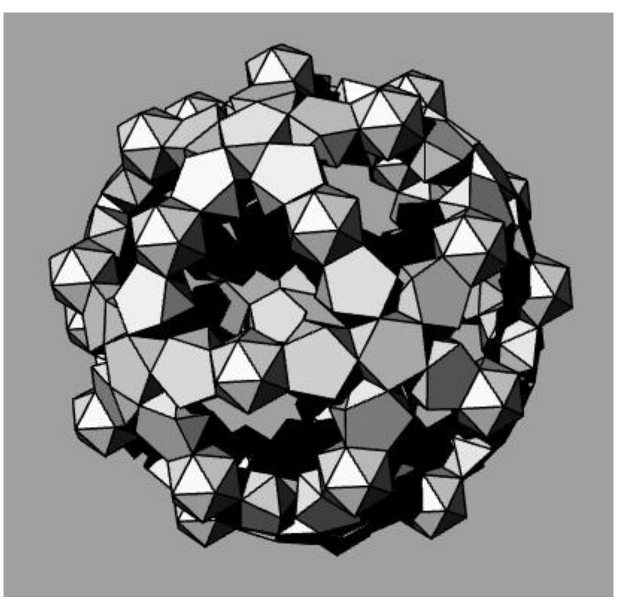

Figure 15: *Icosadodecahedral sphere.*

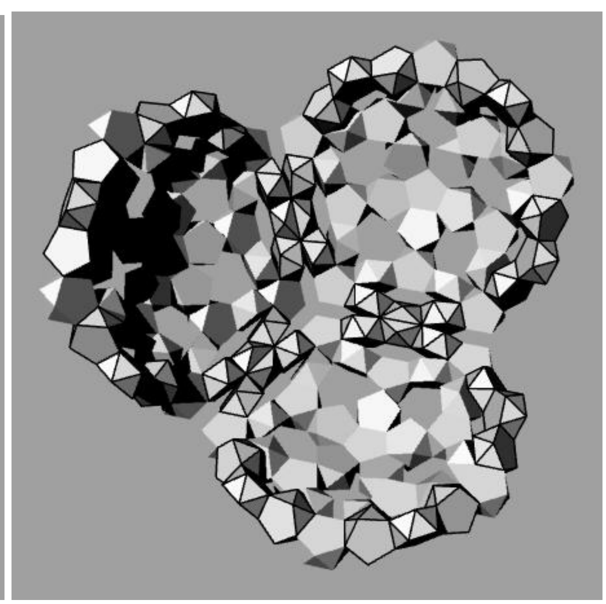

Figure 16: *Rhombicosadodecahedral sphere nr. 1. (fragment).*

By building a sphere which is based on a rhombicosidodecahedron as in Figure 17 we avoid such intersections. Another phenomenon we observe in Figure 17 are the symmetric triangular cycles which can also be used to construct hexagonal infinite knots seen in Figure 18.

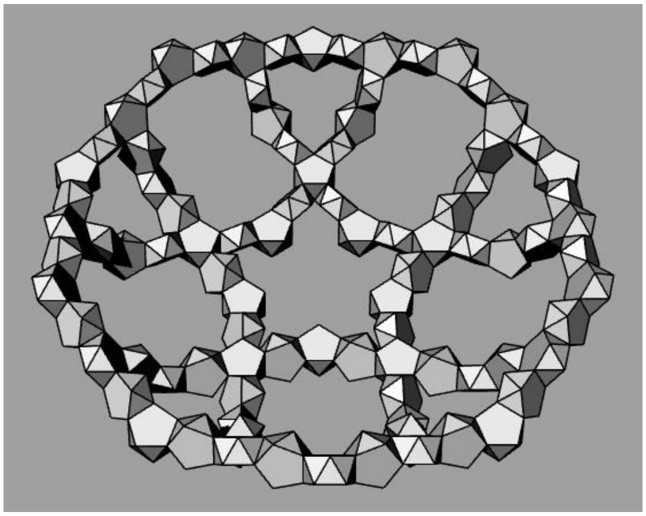

Figure 17: *Rhombicosadodecahedral sphere nr. 2 (fragment).*

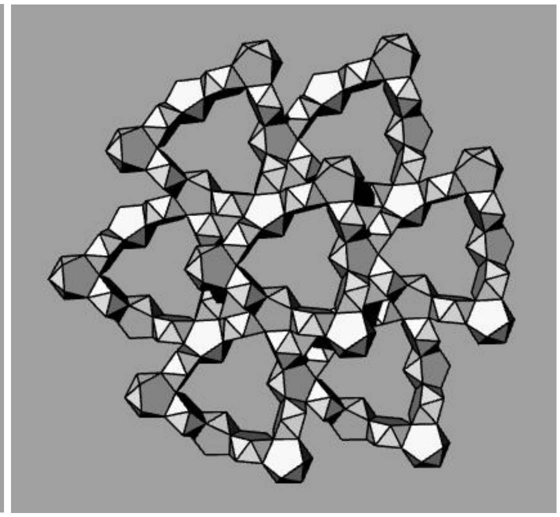

Figure 18: *Hexagonal infinite knot with 5-antiprisms.*

Method of investigation

These results were experimentally obtained using approximations with the help of software and it remains to be determined in the future if these are exact fits. The author is aware of methodological limitations and the exploratory nature of his work. All results are obtained by the use of the software program Rhinoceros [4]. Also the program Great Stella [3] is excellent for quickly making models of this sort as it allows quick and exact face-to-face attachment of polyhedra.

Acknowledgments

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