Creating Self Similar Tiling Patterns and Fractals using the Geometric Factors of a Regular Polygon

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Abstract

A regular polygon with n sides can always be decomposed into isosceles triangles chosen from a set of k non-similar isosceles triangles where k is the integer part of (n-1)/2. It appears that the set displays a property I have, in previous papers, called *preciousness*. This implies that each triangle in the set can be decomposed into assemblies of uniformly scaled triangles chosen from the set. This process can be repeated and forms the basis of recursive tilings and fractals depending upon the details of the process. The value of n has an effect on the symmetry of the design. The relationship between two different regular polygons where the number of sides of one is a factor of the other is also explored. Isaac Newton once famously said that he could see further because he was able to stand on the shoulders of earlier giants. As a theme for decorating the designs I have created pictures of his giants. I have also included pictures of Newton and Einstein as being giants in their own right.

Introduction

Using only the basic properties of the angles in a circle and considerations of symmetry it is easy to prove that a regular polygon can always be dissected into triangles chosen from a set of k non-similar isosceles triangles where k is the integer part of (n-1)/2. The triangles in this set are called the *primary* isosceles triangles. Figure 1 illustrates such n-gon disection for n=3 up to 11. I have numbered the primary isosceles triangles from 1 to k. The numbering may seem arbitary but the number chosen is a measure of the base angle which can be shown to be a multiple of 180/n degrees. For example for the triangle 5-1 the base angle is (180/5)x1 = 36 degrees, for triangle 5-2 the base angle is (180/5)x2 = 72 degrees and for triangle 11-4 we get 65.45 degrees.

The Geometric Factors of a Regular Polygon

With reference to Figure 1 consider the 5-gon and 10-gon.The base angles of 5-1 and 10-2 are (180/5)x1 and (180/10)x2, and both are 36 degrees. Similarly for 5-2 and 10-4 both are 72 degrees. Now assuming the length of the side is the same then triangles 5-1 and 10-2 are identical. The same is true for 5-2 and 10-4, and also true for 3-1, 6-2 and 9-3. I refer to a smaller polygon constructed from the primary isosceles triangle of a larger regular polygon as a *geometric factor* of the larger polygon. Figure 2 shows the complete set of geometric factors for the 30-gon. Conversely, primary isosceles triangles from any regular polygon with n sides can be used to form a regular polygon with m sides, provided n is a multiple of m. Figure 3 shows the formation of the 6-gon from the 12, 18 and 24-gon.

The Precious Properties of Primary Isosceles Triangles.

What is interesting from a self similarity point of view is that for each of the primary isosceles triangles a larger similar triangle can always be produced. This excludes the trivial case where the large similar triangle is composed of 4 copies of itself. The areas of these triangles are the sum of the areas of a subset of the primary triangles. In addition, each of the side lengths of the larger triangle is equal to the sum of those of one or more diagonals of the n-gon. It is possible to prove most of these assertions. I have verified for most regular n-gons where $n < 31$ that every primary isosceles triangle satisfies this enlargement property. As a consequence any design or shape created using the primary isosceles triangles can be enlarged. In addition, the resulting enlargement can be enlarged again using the same process. This procedure can be repeated any number of times. I have called this property *preciousness*. More detailed discussion can be found at [4] [5] [6] [7] [8] [9]. It is always possible to calculate the enlargement factor for any regular polygon. This factor I have called the *precious ratio*. When n=5 this ratio is the golden ratio; When n = 6, the ratio is root three, (see Figure 4). When n = 30, to 3 decimal places, the ratio is 9.514. When the number of sides n of the regular polygon is even then I suspect that the small triangles can be fitted and display mirror symmetry. When the number of sides is a multiple of 3 then I am hoping to find some triangles with a rotational symmetry of order 3. When the number of sides is a multiple of 6 then I am hoping to find some triangles with a rotational symmetry of order 3 and mirror symmetry (see Figure 4).

Precious Triangles and Combinatorics

So far I have managed to prove that any regular polygon can be broken down into primary isosceles triangles. In addition, there is always a set of larger triangles whose area is equal to the sum of the areas of an integral number of these isosceles triangles. The enlarged triangles have side lengths that are the sum of those of one or more diagonals of the n-gon. The fact that the areas and sides correspond are necessary but not sufficient conditions for a precious relationship to exist. I have, however, verified that this is true for the regular polygons from the 3-gon to the 30-gon, (see Figure 6 for the 30-gon). Fitting the small triangles into a large ones was a jigsaw type problem. Manually I started with the border, and filled in the rest. This becomes quite difficult as n becomes large the number of options for the borders and interior rapidly expand. For me, large meant greater than 17 sides. To assist I have written some routines that fit the triangles systematically. I treat each small triangle as a pair of unit vectors or a resultant. To limit the number of options I use the routines on a ”what if?” basis rather than provide a complete final answer. This involved the development of strategies that allowed the combinations of vector operations to be evaluated systematically. This turned out to be very similar to the Lehmer codes or factorial numbers [1] [2] [3] [10] [11] developed in the early 20th century by Derick Norman Lehmer and later by his son Derrick Henry Lehmer. These codes are useful in allowing the problem to be split into several parts allowing parallel processing. In my case this meant running several laptops at the same time. I was also able to generate the codes randomly. One of my early results can be seen in Figure 5. The first stage was to sort out a valid border and then fill in the central section. Figure 5 shows how various alternative solutions may be found. The process was quite slow being written in Vbasic but good enough to get results up to the 30-gon. I was also able to look for alternatives especially searching for solutions with twofold symmetry. These can be clearly seen in Figure 6. Each triangle has been numbered 1 - 14. The numbers refer to numbering used when discussing the geometric factors in the first section. When expanding a design the process can progress using the scheme associated with the factor or the n-gon.

Patterns and Fractals using Geometric Factors and Multiples

Isaac Newton once famously said that he could see further because he was able to stand on the shoulders of earlier giants. I have used, for decoration, Newton's giants with the addition of Newton and Einstein as important in the development of gravitational theory. I have tried to show a cross section of mathematical ideas. Some are fractals i.e. the fractal dimension is between 1 and 2 (see Figure 8 and Figure 9). Some are irregular tilings which, eventually, would fill the plane (see Figure 7 and Figure 10). Some of these tilings have been used to create mosaic type pictures (see Figure 10). Some show the use of factors of large polygons (see Figure 7, Figure 8, Figure 9 and Figure 10). Many of the pictures are derived from portraits constructed from a set of overlapping shapes. The captions should identify the various aspects of the pictures. Most of the pictures or designs are better viewed when much large and in colour.

References

[1] Goodman A.W. and Ratti J.S., Finite Mathematics with Applications - pp. 82-89. 1971 Macmillan - before ISBN

[2] http://www.keithschwarz.com/interesting/code/?dir=factoradic-permutation(as of Nov. 15, 2013)

[3] http://jamesmccaffrey.wordpress.com/2012/09/07/the-factoradic-of-a-number(as of Nov. 15, 2013)

[4] Spencer S J, The Tangram Route to Infinity pp. 52-57 for matrix notation and appendix 6 for eigen values

[5] Spencer S J, Alhambra. Bridges 2003 Proceedings pp. 291-298 An introduction the tiling properties of precious triangles.

[6] Spencer S J, Kansas. Bridges 2004 Proceedings pp. 71-78 The tiling properties of the tangram

[7] Spencer S J,Banff. Bridges 2005 Proceedings pp. 31-36 An introduction to the golden tangram and its tiling properties.

[8] Spencer S J, London. Bridges 2006 Proceedings pp. 73-78 Introducing the precious tangram family.

[9] Spencer S J,Towson. Bridges 2012 Proceedings pp. 337-334 Introducing the use of primary isosceles triangles of regular polygons.

[10] http://en.wikipedia.org/wiki/Lehmer_code(as of Nov. 15, 2013)

[11] http://en.wikipedia.org/wiki/Factorial_number_system(as of Nov. 15, 2013)

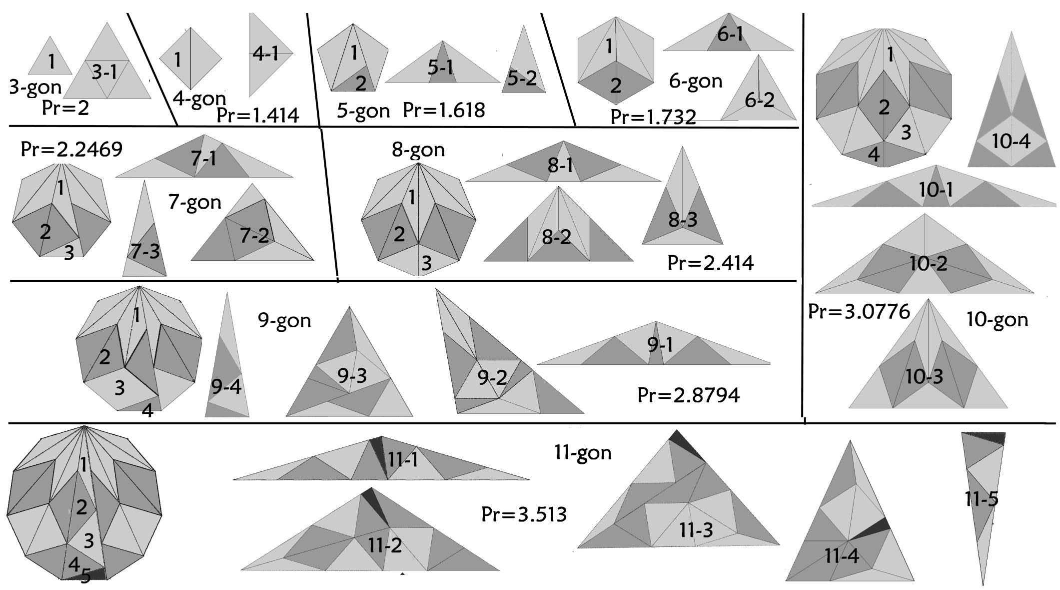

Figure 1 : *The dissection of the regular n-gons into primary isosceles triangles. The enlarged assemblies and precious ratios are also shown.*

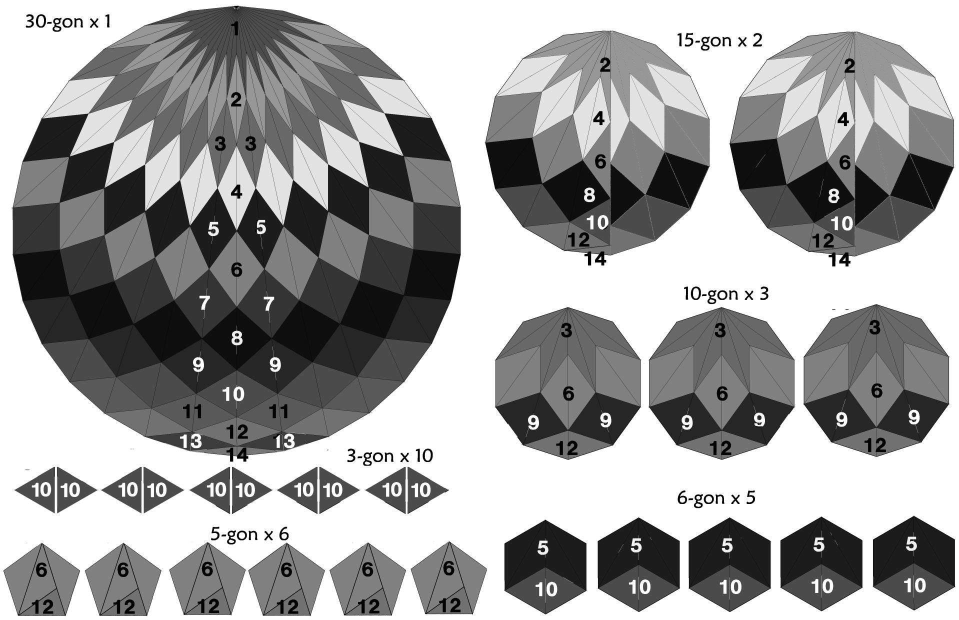

Figure 2 : *The complete set of geometric factors for the 30-gon.*

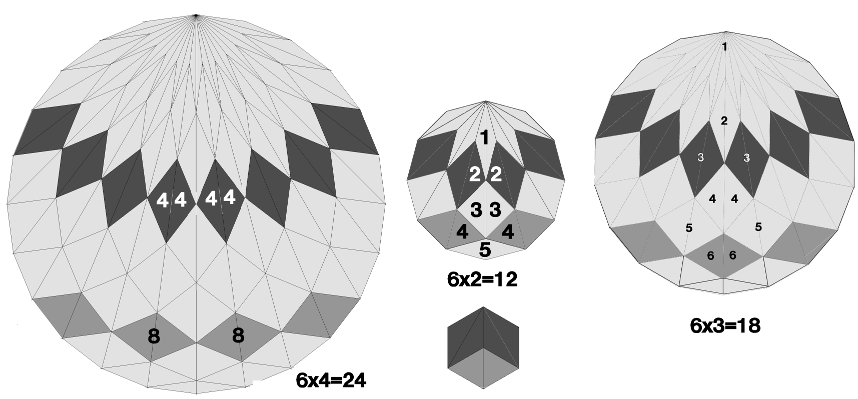

Figure 3 : *Forming the hexagon from the 12, 18 and 24-gon.*

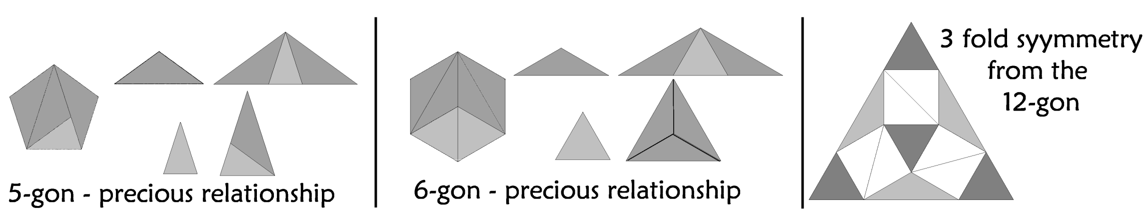

Figure 4 : *Illustration of the precious relationship of the 5-gon and 6-gon and three fold symmetry from the 12-gon.*

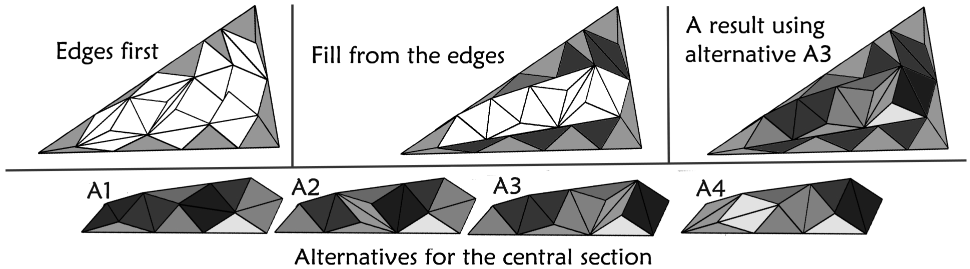

Figure 5 : *These are early results from my jigsaw algorithm fitting the small triangles inside the large. The first stage is to look at the various options for the border and then alternative ways of filling in the central section*

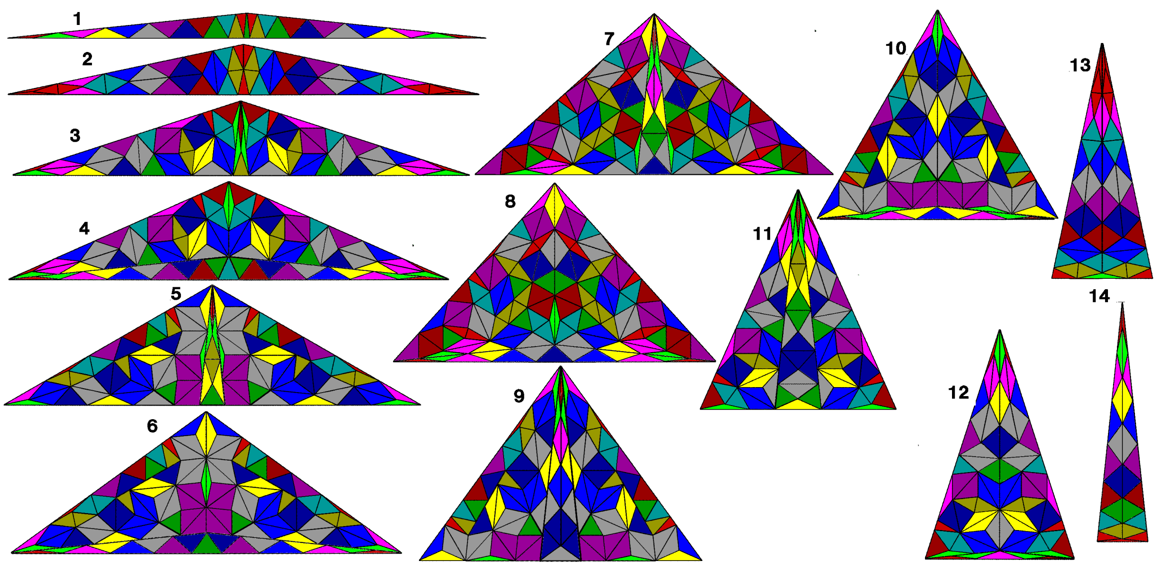

Figure 6 : *These results show the precious expansion of the 14 triangles from the 30-gon. They are numbered 1 to 14. The numbers refer to the numbers in Figure 2.*

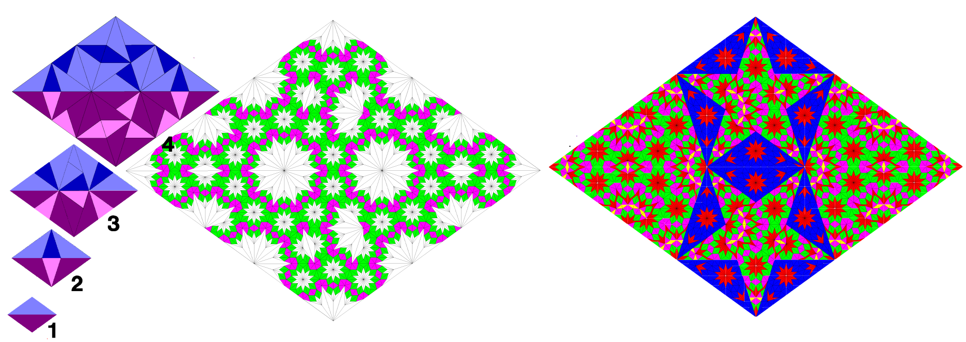

Figure 7 : *These results show the precious expansion of the a design from the 5-gon. Subsequent expansion uses the geometric properties of the 10-gon.*

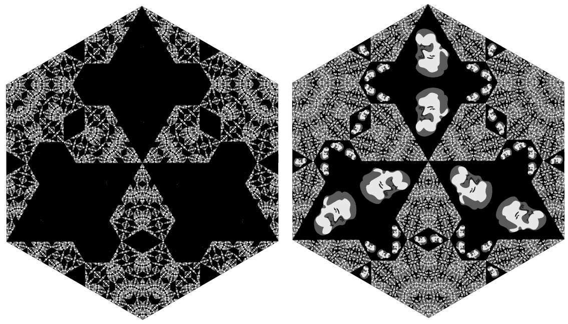

Figure 8 : *These results show the precious expansion of the a design from the 6-gon. Subsequent expansion uses the geometric properties of the 12-gon. The leftmost picture is a fractal. On the right the spaces between the fractal feature my portrait of Galileo.*

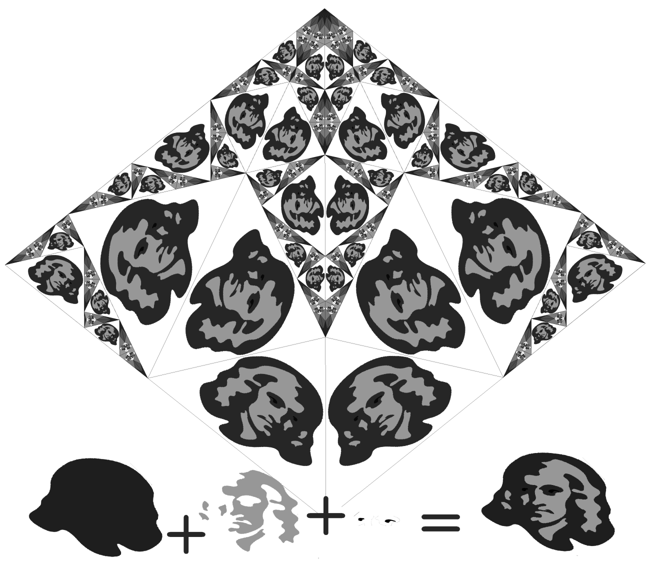

Figure 9 : *These results show the precious expansion of the a design from the 7-gon. Subsequent expansion uses the geometric properties of the 14-gon. The spaces between the fractal feature my portrait of Isaac Newton.*

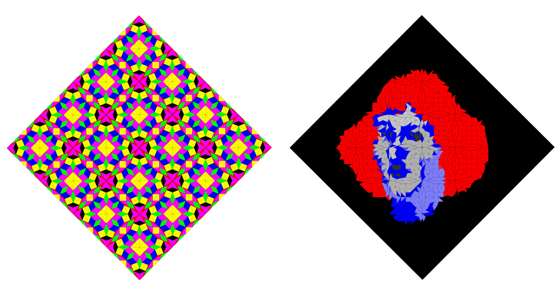

Figure 10 : *These results show the precious expansion of the a design from the 4-gon. Subsequent expansion uses the geometric properties of the 16-gon. The leftmost design is the first few stages of an infinite tiling in the plane. The rightmost is a mosaic type picture derived from a my portrait of Copernicus.*