Elevations and Stellations

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Abstract

Stellation is a well known operation which you can apply to a polyhedron. About a century before this operation was defined by Kepler, Luca Pacioli and Leonardo da Vinci had published the book "La Divina Proportione" in which another operation, called Elevation, was described. Both operations have much in common, but there are also many differences. Comparing both operations inspired me to define a new one, which I called Edge Elevation. This led to interesting artistic constructions.

1. Introduction

1.1. Elevation. In their book "La Divina Proportione" [2] Luca Pacioli and Leonardo da Vinci introduced the elevated versions of all of the Platonic polyhedra as well as of some of the Archimedean polyhedra. The Platonic solids as well as their elevations as they are drawn by Leonardo da Vinci are shown in Figure 1 and 2.

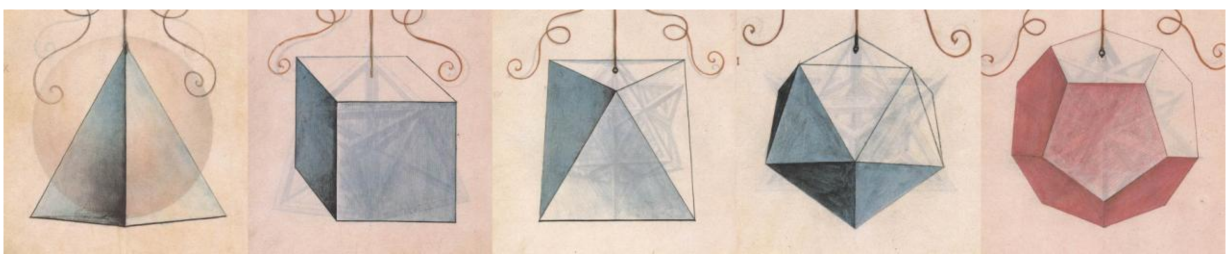

Figure 1: *The Platonic Solids*

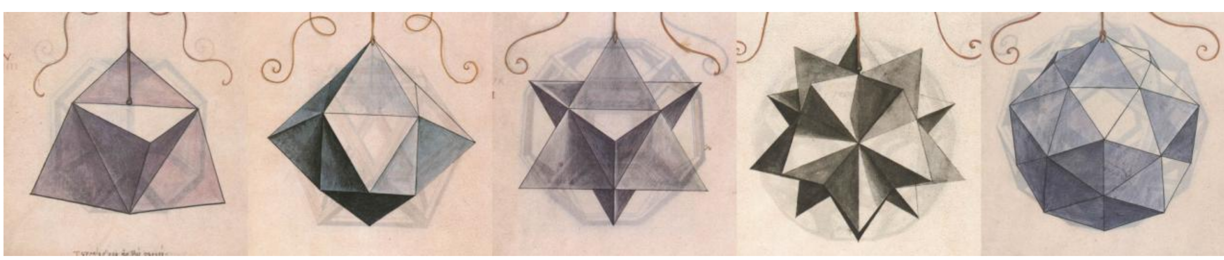

Figure 2: *Elevations of the Platonic Solids*

What exactly is an elevated version of a polyhedron? In "La Divina Proportione" [3],chapter XLIX, paragraph XI.XII, Pacioli describes the elevated version of the cube as follows: "… it is enclosed by 24 triangular faces. This polyhedron is built out of 6 four-sided pyramids, together building the outside of the object as you can see it with your eyes. And there is also a cube inside, on which the pyramids are placed. But this cube can only be seen by imagination, because it is covered by the pyramids. The 6 square faces are the bottom faces of the 6 pyramids.".

So in total this object is composed of 24 equilateral triangular faces plus 6 hidden square faces, as can be seen in the exploded views in Figure 3.

Figure 3: *Exploded views of the elevated cube and octahedron*

About the "Octocedron Elevatus", Pacioli writes (Chapter L, paragraph XIX.XX): "And this object is built with 8 three-sided pyramids, that can be seen with your eyes, and an octahedron inside, which you can only see by imagination.". This means that the object is composed of 32 equilateral triangular faces of which 8 are hidden.

1.2. Stellation. Pacioli doesn't give a real definition of "Elevation" but his descriptions are very clear. In 1619, about one century later, Kepler defined "Stellation" for polygons and polyhedra as the process of extending edges or faces until they meet to form a new polygon or polyhedron [1].
Following the definition of Kepler, the stellation of the octahedron has the same number of faces as the octahedron itself since the eight triangular faces are extended to eight bigger triangular faces. These large triangular faces intersect each other, forming the so-called Stella Octangula or Kepler Star. As we can see in M.C. Escher's print, this object is in fact a combination of two interpenetrating tetrahedra (Figure 4).

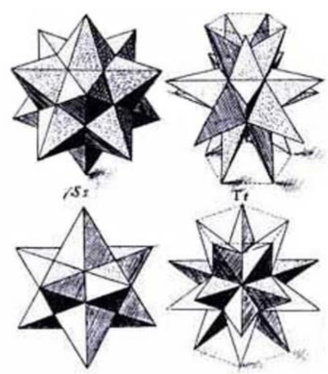

Figure 4: *M.C. Escher - Two worlds.* **Figure 5**: *Kepler's Stellations.* **Figure 6**: *M.C. Escher - Gravity*

2. Elevation versus Stellation

2.1. Difference. There is a fundamental difference between the elevation of the octahedron and the stellation of the octahedron: the total number of faces of the elevation is 32, whereas the number of (intersecting) faces of the stellation is 8. And yet there is another important difference. Kepler, in his definition, talks about a process. When we read the descriptions of Pacioli, he is just talking about the final result. We can however redefine "Elevation", introducing the process as follows: Elevation for polyhedra is the process of pulling each midpoint of all of the faces outwards until the triangles formed by those midpoints with two adjacent vertices of the original face form are equilateral.

A generalization can be made by not demanding that the triangles must be equilateral. With this definition Pacioli's elevated dodecahedron can be seen as a step in between the dodecahedron and Kepler's first stellation of the dodecahedron. In the print "Gravity" by M.C. Escher (Figure 6) parts of the star shaped-faces are removed, which gives us a good look at the construction. Escher writes about this print: "On each of the twelve faces we can see a monster of which the body is captured under a five sided pyramid.". Which comes close to Pacioli's description of the elevated polyhedra. Using the new definition of elevation, we can compare the process of development from dodecahedron to the elevated version with the development from dodecahedron to Kepler's stellation. In the pictures (Figure 7 and 8) only 6 faces (the upper part) of the dodecahedron are shown. In the elevation process, the third step shows the object that is published in "La Divina Proportione", but when you continue the process of elevation you can end up at the state that is similar to the stellated dodecahedron, since in this state, five triangular faces of the elevated figure are coplanar with an original face of the dodecahedron.

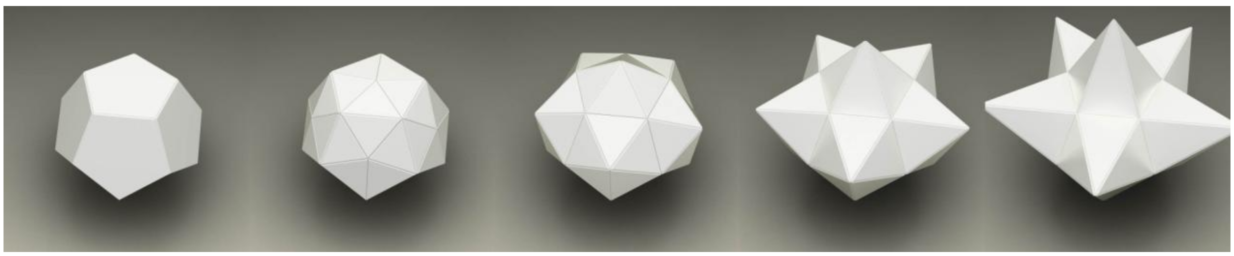

Figure 7: *Elevation.*

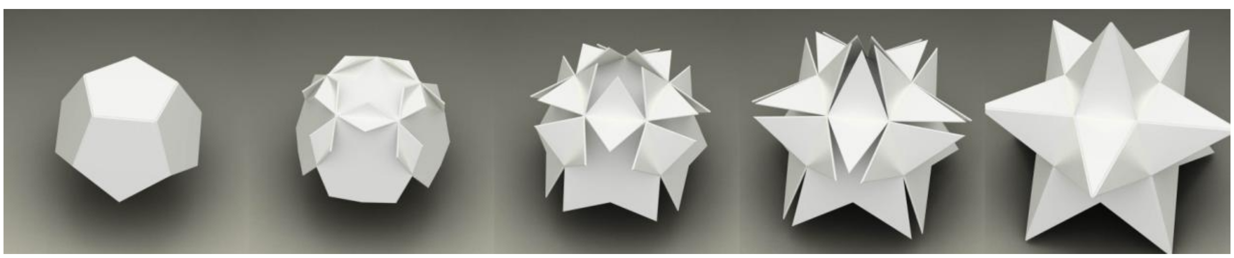

Figure 8: *Stellation.*

The big difference between the two final objects is the number of faces counted. In the elevated version we still have the pentagonal faces inside, so both objects are double layered.

2.2. Second stellation. Starting with the icosahedron we can extend the triangular faces to create a stellated icosahedron. Four steps of this dynamic process are shown in Figure 9. In the process of stellation this is the first position in which we get a new polyhedron (Figure 10).

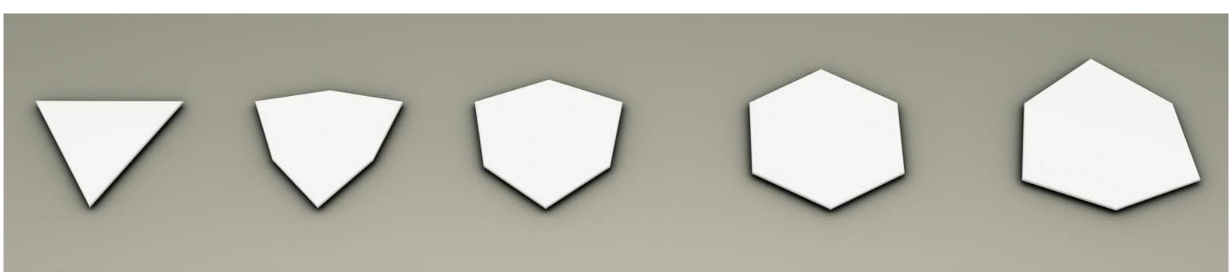

Figure 9: *Development of the shape of the faces for the first stellation of the icosahedron.*

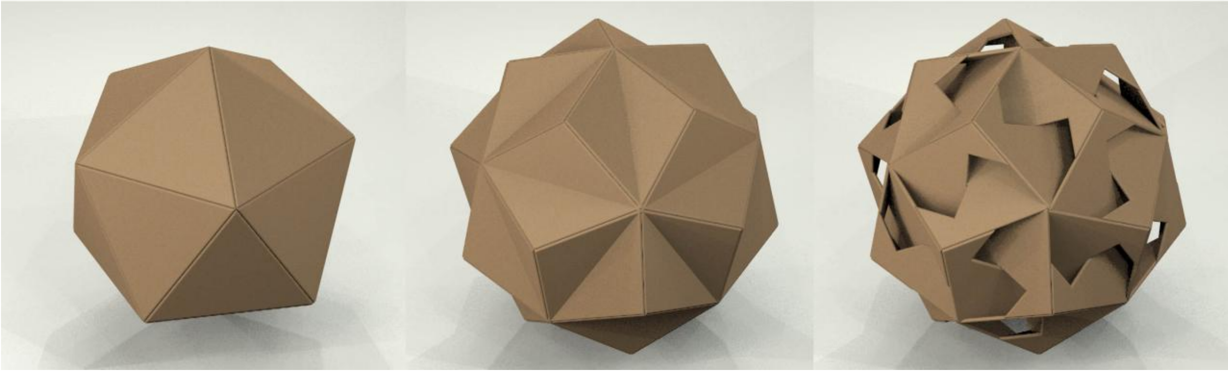

Figure 10: *First Stellation.*

But we can continue the process and extend the faces a little further (Figure 11) The second position where we have a polyhedron again is shown in Figure 12.

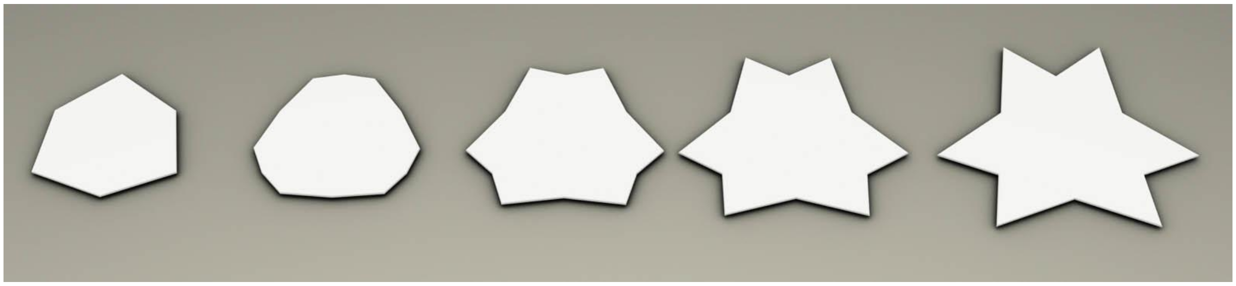

Figure 11: *Development of the shape of the faces for the second stellation of the icosahedron.*

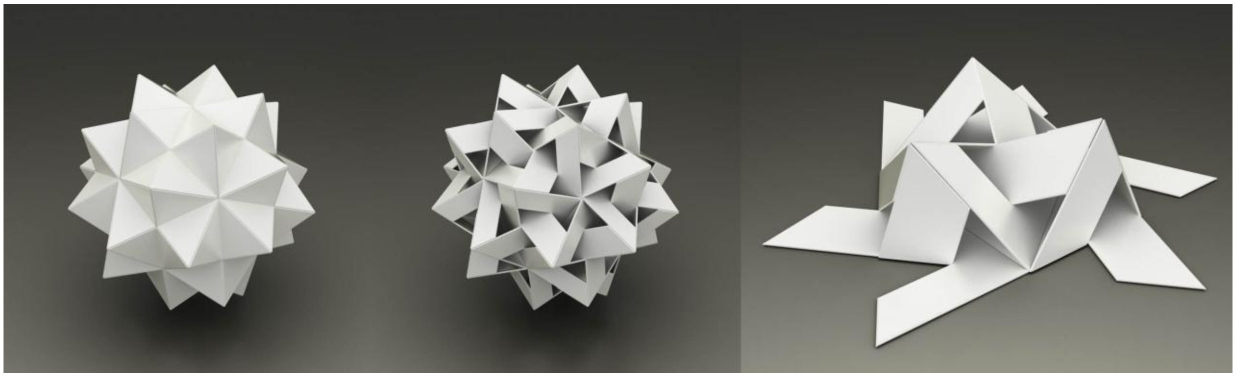

Figure 12: *Second stellation of the icosahedron.*

Just like in Escher's print "Gravity" I have removed parts of the faces to get a good idea of the construction. I was fascinated by the interesting structure that appeared now on top of all faces. Not one pyramid but an intersection of 3 somewhat deformed pyramids. So one step more in the stellation process leads to very interesting structures. The question now arose if something similar would be possible with Elevation. Can we define second elevation? And what kind of objects can we expect then?

2.3. Second elevation. I used the generalized elevation process defined in section 2.1 and applied this to the octahedron, elevating the midpoint of each face of the octahedron until the new shape was similar to another polyhedron. The first polyhedron obtained this way was the rhombic dodecahedron.
On this polyhedron we can apply the same process again and call the next result a second elevation of the octahedron. We stopped at the point where we got coplanar sets of elevated faces on the faces of the rhombic dodecahedron (Figure 13).

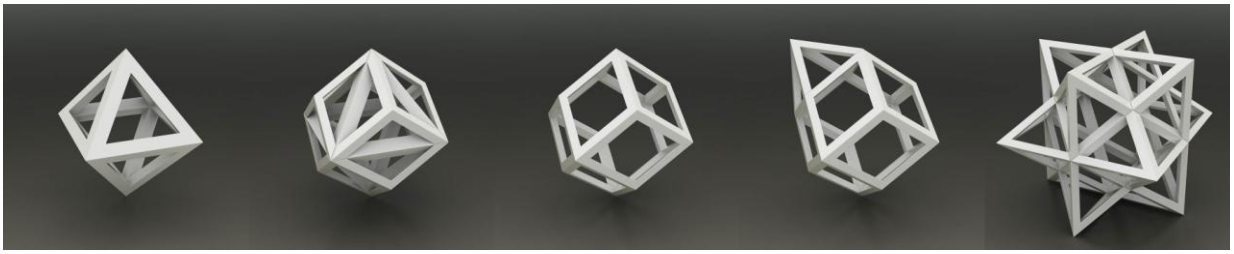

Figure 13: *First and second elevation of the octahedron.*

The right most figure now appears to be the same as one of the polyhedra Escher constructed (see Figure 14) and used in his print “Waterfall” (Figure 15).

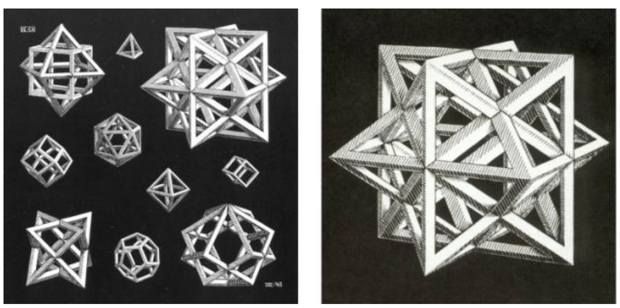

Figure 14: *Skeleton of Escher’s polyhedron.*

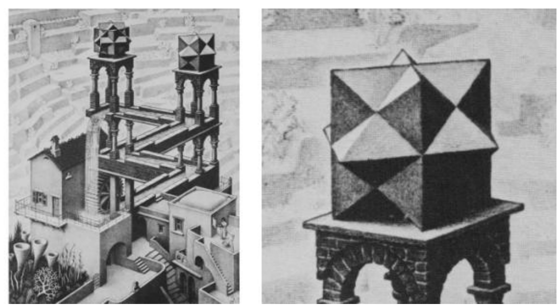

Figure 15: *Escher’s polyhedron in “Watervall”.*

3. Edge Elevation

3.1. Edge Elevation. Studying the end figure a little more brought me to the conclusion that you also can see this as an octahedron with a somewhat deformed rhombic pyramid on each of the edges. Twelve pyramids surround the octahedron. That means that we can reach the same end figure by elevating the midpoint of each of the edges. So we can define a new transformation: “Edge Elevation”, in which the midpoint of each edge is connected to the midpoint of the faces that meet that edge and then the edge midpoint is pulled outward, stretching its connection to form a pyramid of four triangles.

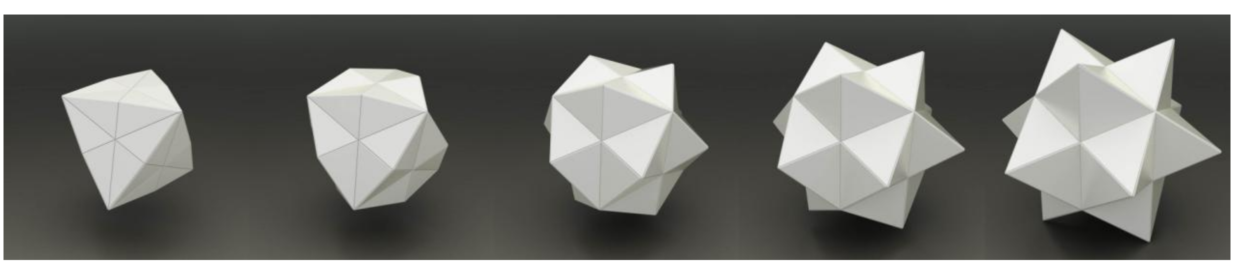

Figure 16: *Development of the shape of the faces of the second stellation of the icosahedron.*

Figure 17: *Development of the shape of the faces of the second stellation of the icosahedron.*

Figure 16 illustrates the process. Figure 17 shows the same process but now uses Escher's way of opening up the structure as in his print "Gravity" (Figure 6).

3.2. Tetrahedron and Icosahedron. When we apply edge elevation to other polyhedra such as the tetrahedron and icosahedron we will see developments of these objects as in Figure 18 and 19. The final state of the development of the edge elevated icosahedron is similar to the second stellation of the icosahedron. In the case of the tetrahedron we end up with a new object.

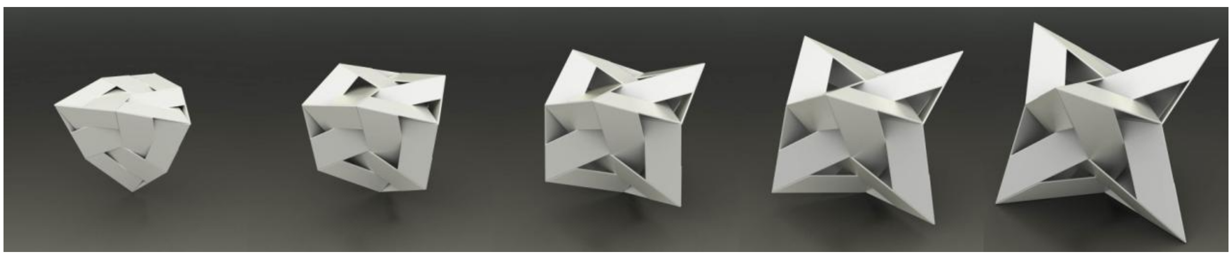

Figure 18: *Development of the shape of the edge elevated tetrahedron.*

Figure 19: *Development of the shape of the edge elevated icosahedron.*

4. Construction

4.1. Construction Elements. The final state of the edge-elevated icosahedron looks similar to the second stellation of the icosahedron but there is big difference. When we analyze the construction of both objects we will see that their faces are different. At left in Figure 20 is one of the faces of the second stellation of the icosahedron, at the right two connected faces of the edge-elevated icosahedron.
To build models of the edge-elevated polyhedra I laser-cut the faces in pairs with a folding line on the common edge. The paper models appeared relatively simple to build.

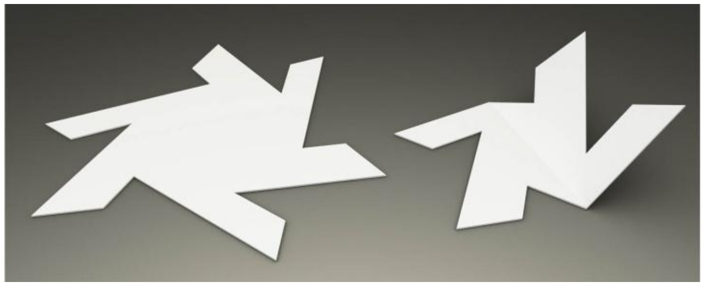

Figure 20: *Faces of the second stellation of the icosahedron and of the edge-elevated icosahedron.*

To start building the model of the edge-elevated icosahedron I first connect three pairs of faces as you can see in Figure 21. Both the top and the bottom are shown.

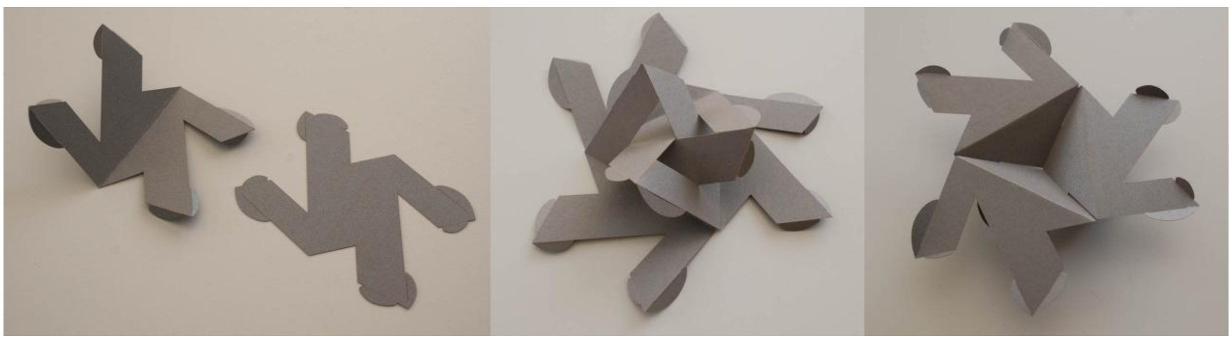

Figure 21: *Construction elements – Three connected elements (top view) – Three connected elements (bottom view).*

As you can see, when you connect more elements, the structure at the outside becomes quite complicated, while the inside of the polyhedron is a nice diamond pattern (Figure 22,23).

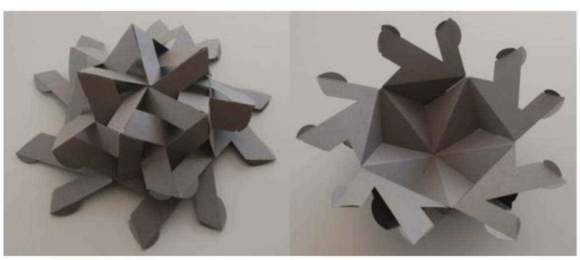

Figure 22: *Ten pairs of faces are connected.*

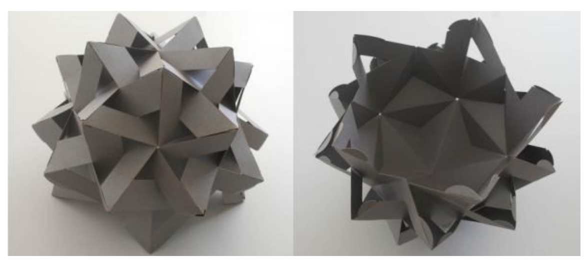

Figure 23: *Twenty pairs of faces.*

5. Examples

5.1. Geodesic Spheres. The method of elevating the midpoint of each of the edges of a polyhedron can be applied on many different objects. For the examples I have limited myself to objects built with only triangles. The triangles don't have to be equilateral, as can be seen in the examples of the geodesic spheres in Figure 24, 25 and Figure 26, 27. Because the construction is double layered, it is quite strong and could very well be used for dome constructions.

Figure 24: *Geodesic sphere - inside.*

Figure 25: *Geodesic sphere -outside.*

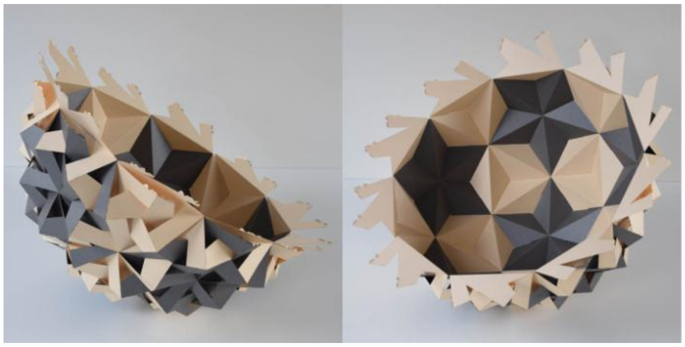

Figure 26: *Geodesic sphere.*

Figure 27: *Half a sphere: dome.*

5.2. Cylinders. As a final example I would like to show a cylindrical construction (Figure 28, 29). The basic polyhedron is a helical deltahedra. I think we may conclude that the "edge elevation" is an interesting new operation with which we can generate interesting constructions.

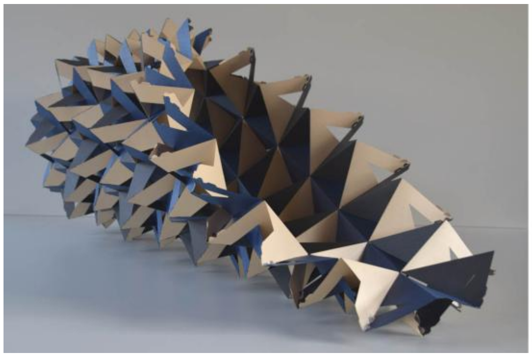

Figure 28: *Edge-elevated cylinder.*

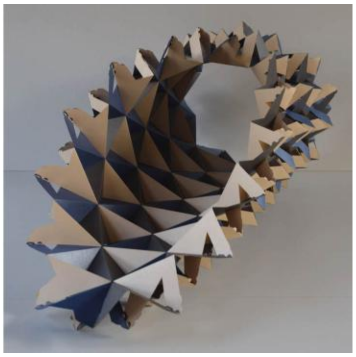

Figure 29: *Edge- elevated cylinder.*

References

[1] Peter R. Cromwell, *Polyhedra*, Cambridge University Press, UK, 1997.

[2] Luca Pacioli – Leonardo da Vinci, *La Divina Proportione,* 1509, Ed. Akal, S.A., Madrid, 1991

[3] Luca Pacioli, *Divina Proportione: Die Lehre Vom Goldenen Schnitt,* 1509, Ed. Carl Graeser, Wien, 1896.

[4] Red. J.W. Vermeulen, *Het oneindige, M.C. Escher over eigen werk*, Meulenhoff, Amsterdam, 1986.

[5] Magnus J. Wenninger, *Polyhedron Models,* Cambridge University Press, UK, 1971.