The *Kinochoron*: A Manipulable Wire Model of the 16-cell

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Abstract

4-dimensional polytopes can be visualized by projecting them to our 3-space. The projections are usually represented either with computer-generated animations or with rigid objects. This paper deals with the possibility of representing a polychoron with a kinetic 3-dimensional model. A stereographic projection of the 16-cell is constructed with six hexagram links of wire forming Chinese crosses at the eight vertices. The symmetry of the model permits a manipulation similar to the 'hyper-rotation' in animations representing stereographic projections of polychora. Playing with the design offers a hands-on approach to the study of 4-dimensional geometry and the hypersphere.

Introduction

4-dimensional space is an abstract concept of adding an extra spatial dimension perpendicular to our three dimensions of length, height and width. Research on the properties of such an environment is made possible by generalizing the geometric principles acquired by studying more familiar spaces of lower dimensions. Well-known 4-dimensional shapes are the six convex regular polytopes called polychora. These shapes can be represented in our 3-dimensional environment as geometric projections. [2]

Of the different projection methods two are most common: orthographic and stereographic projection. The animation film *Dimensions* [1] shows how orthographic projection preserves straight edges but distorts the angles at the vertices, whereas stereographic projection preserves the angles but bends the edges to curves. The film suggests that stereographic projection is the most efficient way to represent polychora because it does away with the inevitable overlap of the individual cells in orthographic projection. When the polychoron in 4-space is set into rotating motion, the projection point travels through the cells and produces a mesmerizing visual effect of 'hyper-rotation' in the stereographic projection output.

On the other hand, a 2-dimensional medium like computer-generated animation lacks the tactile and plastic availability of a 3-dimensional object. Physical models of projected polychora have included orthographic projections in the form of various ball-and-stick designs such as *Zometool* constructions [3], and 3D-printed prototypes of stereographic projections [7]. These rigid objects however fail to communicate those structural relations of a polychoron that are further demonstrated in animations by the movement of the projection point with respect to the shape being projected.

The topic of the paper at hand is the *Kinochoron* design – a stereographic projection of a 16-cell in a physical model that is capable of hyper-rotations evident in the animations of the same projection method. The design is, to my knowledge, the first kinetic model of a 4-dimensional polytope. Although an object like this hasn't been designed earlier, it has been speculated about in a science fiction story *Mimsy Were the Borogoves* [6]. Written by Henry Kuttner and Catherine L. Moore under the pen name of Lewis Padgett, the story portrays two children who after training manage to move along the fourth dimension of space. The knowledge required for the feat is mediated by Lewis Carroll's nonsense poem *Jabberwocky* and a foldable puzzle toy composed of interlocked wires and beads in the form of a hypercube.

Considering only the frameworks of the edges and excluding all 2-dimensional faces, as is usually the case when designing physical models of the polychora, the realization of hyper-rotation in a 3-dimensional object still seems – at first glance – to require stretching material to be used on the edges. This problem is eliminated by choosing to model the 4-dimensional cross-polytope, the 16-cell, the cruciform vertices and consecutive edges of which can exploited to achieve a kinetic structure.

Linking the edges

The 16-cell has 8 vertices, 24 edges, 32 triangular faces and 16 tetrahedral cells. When the 16-cell is stereographically projected into 3-space choosing the center of one of the cells as the projection point, the edges of the polychoron are arranged in six circles consisting of four edges each. This means that the edges can pass parts of their length to their neighbors belonging to the same circle through the vertices without changing the radius of the circle loop.

This results in some special requirements for a vertex, as it has to be designed so that the edge loops can move freely through it in three perpendicular directions without falling off of the vertex completely. This is accomplished by duplicating and linking the edge loops appropriately, so that a vertex takes the form of a 'Chinese cross' puzzle (figure 1). Giving each of the vertex-to-vertex segments a 3/4 twist further reinforces the vertices and edges (figure 2). Hence each two-strand edge loop twists three times in total and acquires the form of a hexagram link (figure 3).

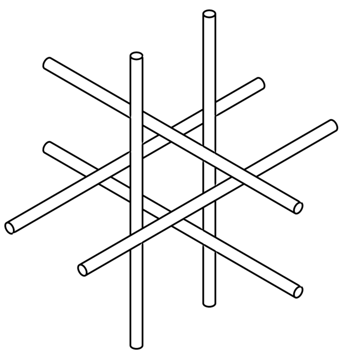

Figure 1: *A vertex, 'Chinese cross'*

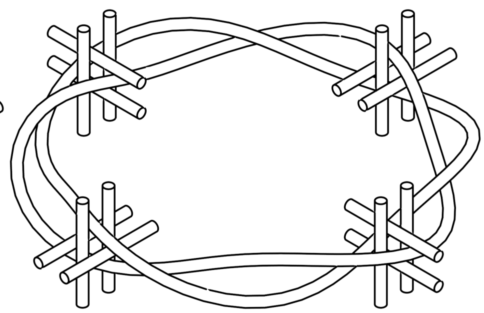

Figure 2: *3/4 twists on four edges between four vertices on an edge loop*

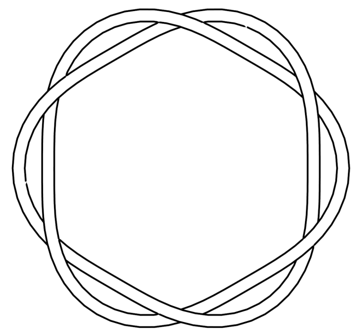

Figure 3: *A loop of four edges, hexagram link*

The edge loops are easily constructed from steel wire – a material suitable for the purpose of manufacturing kinetic models, as it gives in for substantial bending without losing its preference to straighten up. Dmitri Kozlov has shown how form-finding, resilient filaments like steel wire can be knotted to make manipulable 3-dimensional shapes [4, 5]. The final design, the *Kinochoron* (figure 4), consists of steel wires encased in thin brass tubes. This manufacturing technique allows the use of butt joints that are essential for easy sliding of the edges.

The *Kinochoron*

The completed design consists of 12 circular hoops. Each hoop forms a hexagram link with its mate, and a simple Hopf link with each of the 10 hoops belonging to other hexagrams. Each of the six hexagrams comprises four edges of the 16-cell, resulting in a total of 24. All of the hexagrams take part in defining each of the 16 tetrahedral cells with one edge, and each edge takes part in defining four cells. All the hexagrams are joined together with eight 'Chinese crosses', corresponding to the eight vertices of the 16-cell. At each vertex, six edges from three different hexagrams meet. In terms of knot theory, the entire design is a link of 12 components.

The *Kinochoron* represents the edges of a stereographic projection of the 16-cell into 3-space where the projection point is the center of one of the cells. By examining the edges, we can imagine the spheres on

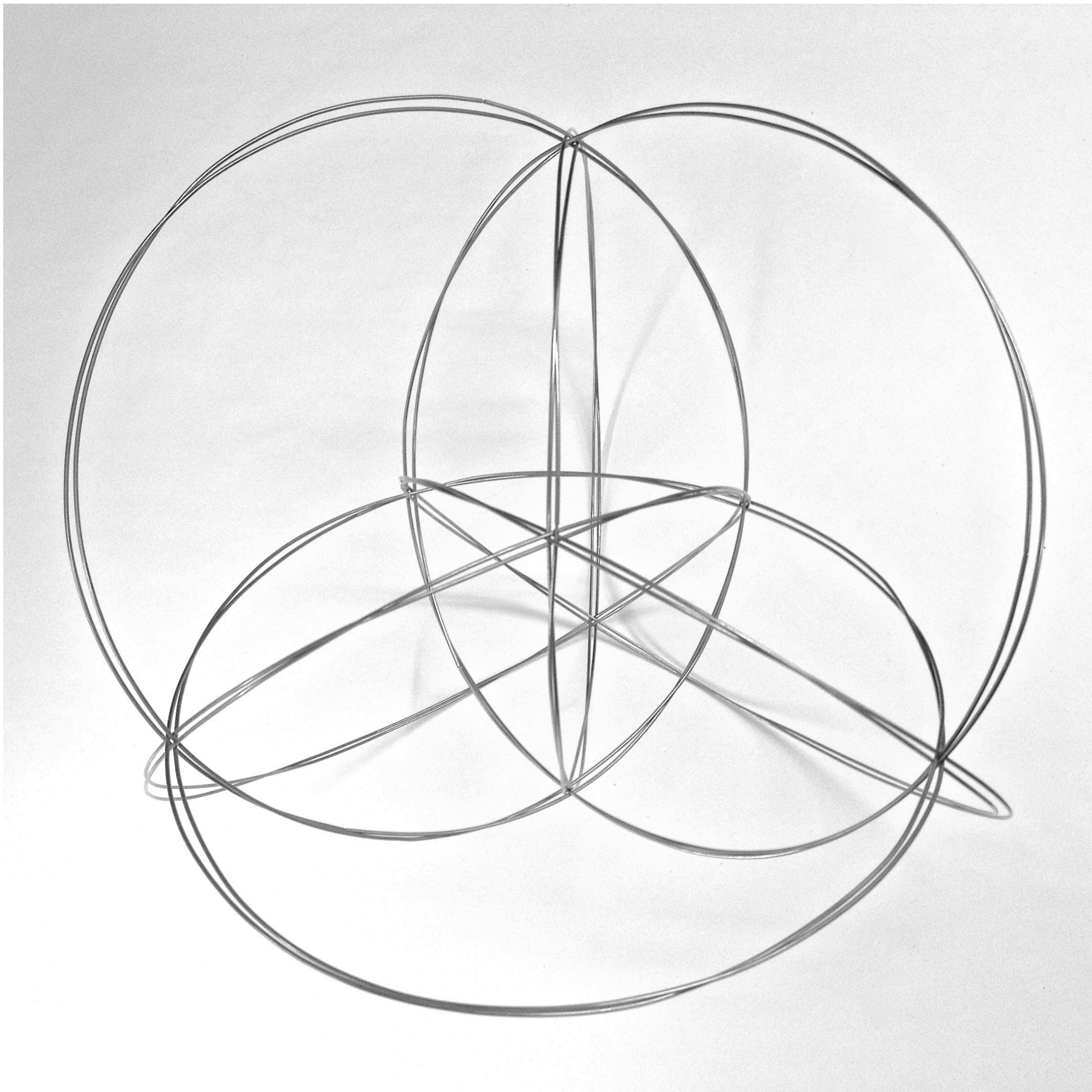

Figure 4: *The 'Kinochoron'*

which they lie and the spherical triangles they define. By combining these triangular faces, we can visualize the tetrahedral cells. In the *Kinochoron*, the tetrahedral cells come in five distinct types of shapes (figure 5). These tetrahedral types are aligned on five nested layers with varying quantities, and have convex and concave faces as follows:

Type:	*A*	*B*	*C*	*D*	*E*
Layer:	1	2	3	4	5
Quantity:	1	4	6	4	1
Convex faces:	4	3	2	1	0
Concave faces:	0	1	2	3	4

Table 1: *Tetrahedral components*

The type *E* tetrahedron is a projection of the cell containing the projection point, and has its center projected at infinity. This causes the cell to be perceived as from the inside. In the entirety of the structure, a pair of tetrahedral cells can share a face, an edge or a vertex. If the two tetrahedra are not connected by any of these 0-, 1- or 2-dimensional boundaries, they are situated on the opposite sides of the 16-cell.

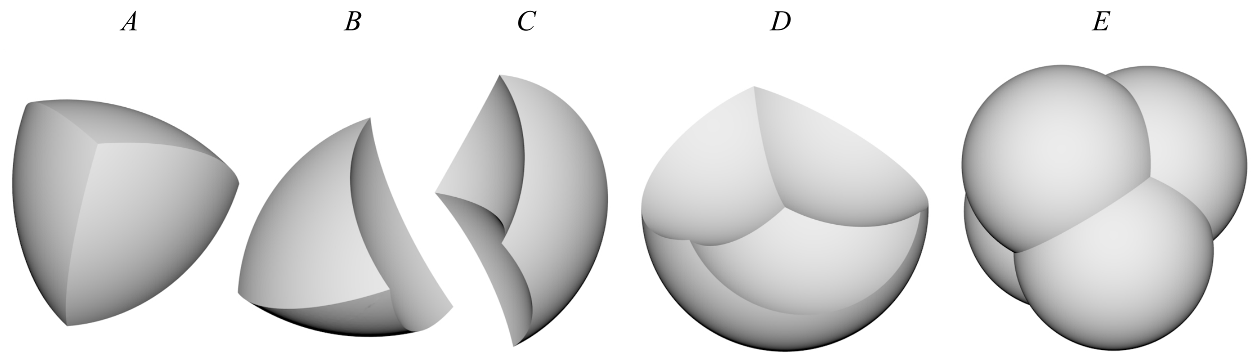

Figure 5: *The five distinct types of tetrahedral cells in the* Kinochoron
(not in correspondence with the actual respective sizes)

As a 3-dimensional object, the *Kinochoron* exhibits the 2- and 3-fold rotations of a tetrahedron that can be perceived just by turning the object. In addition to these symmetries, the kinetic interweave of the wires in the design allows a manipulation (figure 6) that makes all the tetrahedral cells acquire new positions and shapes in the structure. This movement is analogous to the hyper-rotation in stereographic projection where the projection point follows the edges of the dual hypercube, i.e. the projection point travels from the center of a cell to the center of the neighboring cell through the center of the face they share. The twists on the edges force the neighboring vertices and cells into movement, and all the parts of the 16-cell follow the manipulation faithfully to their designated configurations while preserving their respective relations. By pulling the opposite edges of a type *C* tetrahedron, it is possible to perform two cell-to-cell rotations with a single movement and turn a tetrahedron of type *C* into a tetrahedron of type *E*. While manipulating the model, we can imagine ourselves flying from cell to cell through the invisible triangular faces. The cells expand in turns fitting us and the entire surrounding universe inside them while the whole structure is observed in front of us as if seen through an extraordinarily powerful fisheye lens.

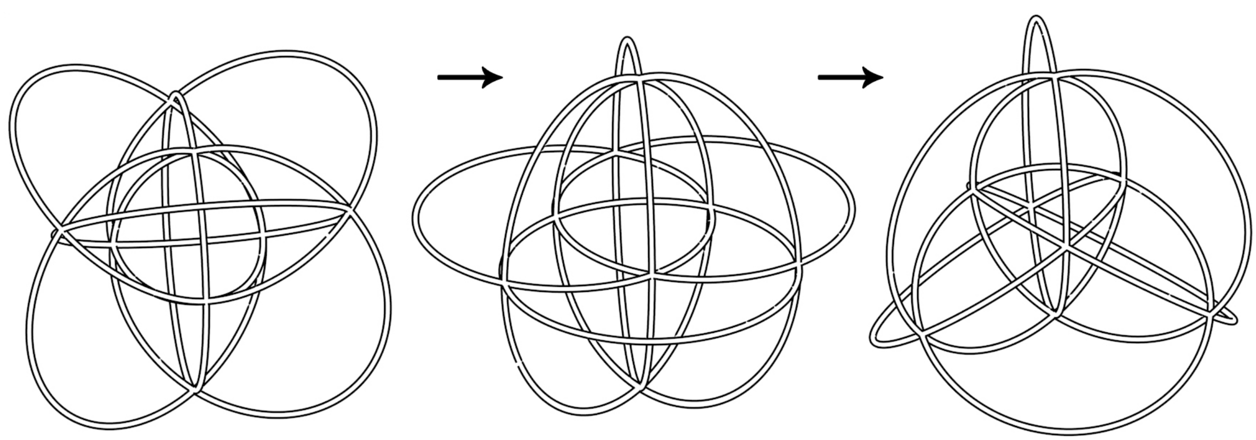

Figure 6: *Cell-to-cell hyper-rotation of the* Kinochoron *through a face*

Although the manipulation of the model resembles the hyper-rotation evident in computer-generated animations of stereographic projections, it must be noted that the *Kinochoron* doesn't respect the precise geometry of an accurately executed stereographic projection that would project the edges into specific lengths. Because a stereographic projection in which the projection point is in the center of a cell projects all the circles to same size, the vertices of the *Kinochoron* can be adjusted along the edges to accurately represent this projection. Usually the unstable structure of the *Kinochoron* depicts only the overall shape of the projection. Moreover, during the hyper-rotation, the hexagrams of the *Kinochoron* bend slightly to go around each other (see figure 6, middle). In an accurate projection, the radii of the corresponding circles would change to some extent.

When all 12 hoops are pushed together, the model collapses flat for easy transport. Consequently it can be represented as a braid, an ornamental depiction of which is given in figure 7. With some effort it is possible to find all 16 tetrahedral cells in the picture. Figure 7 shows also how the numbers from one to eight can be distributed 'magically' to the vertices of the 16-cell. The magic sum of each of the edge loops is 18, and the configuration produces also the sum of 27 with each set of six vertices sitting on the same spherical surface.

Conclusion

In the context of a descriptive geometry course I teach to art students, I have given lectures on the fourth dimension of space. During these talks I have experienced that although the *Kinochoron* may not enable the students to escape our 3-dimensional space like the hypercube of Padgett's story, it is useful in explaining the structure of the 16-cell and demonstrating how on the surface of the hypersphere, as on the actual polychoron in 4-space, all of its 16 tetrahedra are identical.

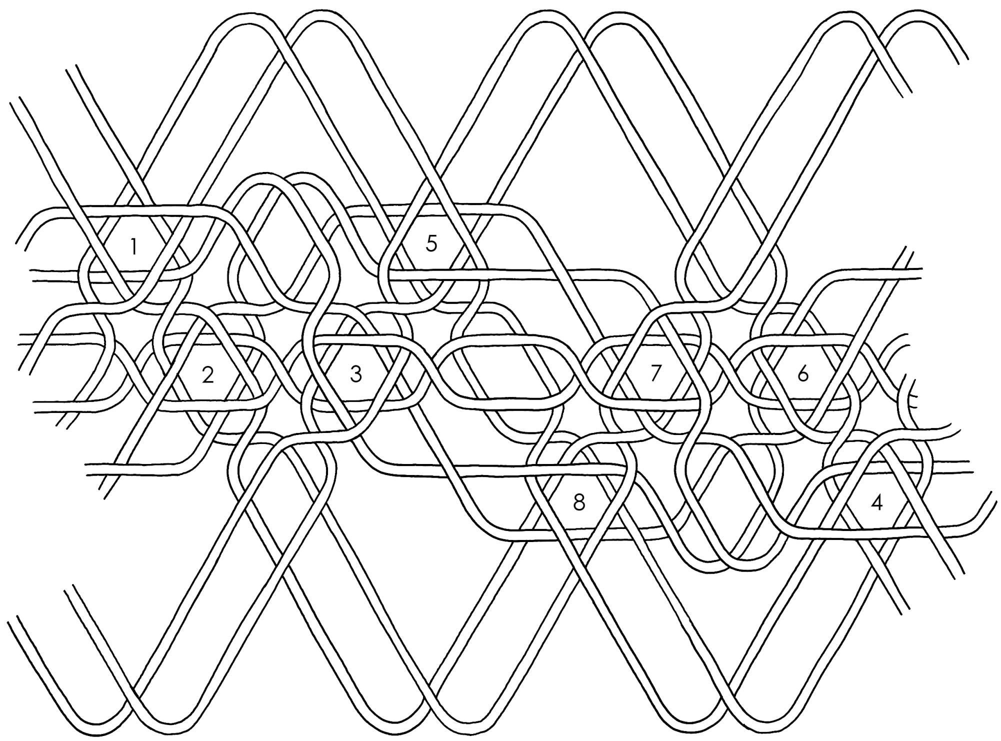

Figure 7: *The interweaved wires of the* Kinochoron *as a braid*

The successful realization of hyper-rotation in the *Kinochoron* bodes well for further investigations into physical and hyper-rotating representations of 4-dimensional polytopes. Considering other polychora, it is evident that alternative weavings have to be devised. One such solution for the 5-cell is presented in figure 8, the vertex interlace of which actually results in a design resembling a truncated 5-cell.

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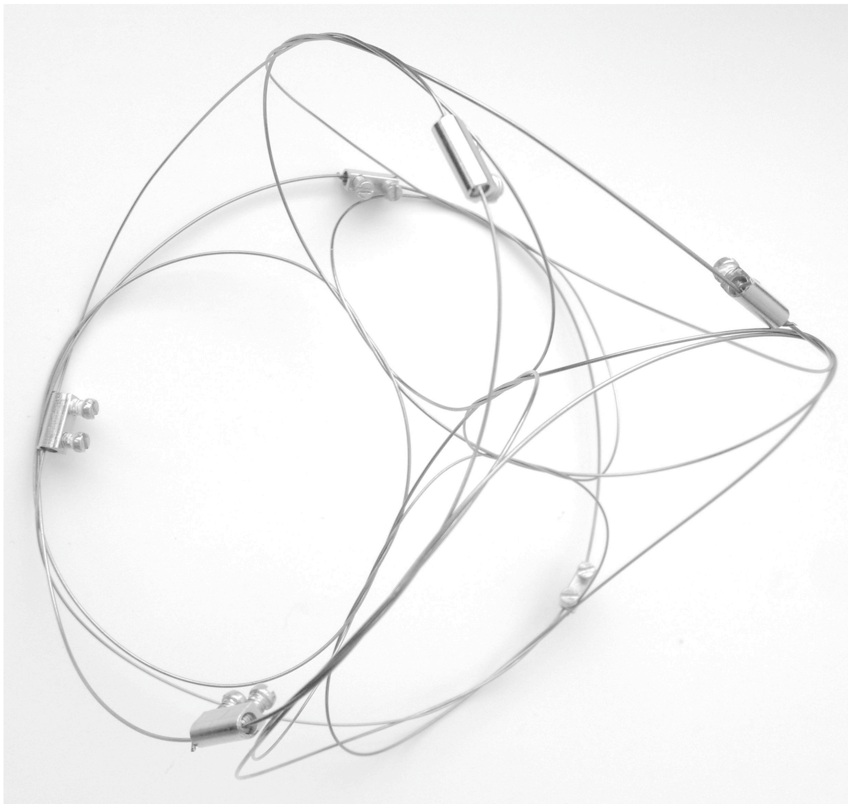

Figure 8: *A hyper-rotating wire model of the 5-cell (truncated 5-cell)*

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