Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture

The Meta-golden Ratio Chi

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Abstract

Based on artistic interpretations, art professor Christopher Bartlett (Towson University, USA) independently rediscovered a mathematical constant called the ‘meta-golden section’, which had been very succinctly described 2 years earlier by Clark Kimberling. Bartlett called it ‘the chi ratio’ and denoted it by χ (the letter following ϕ, the golden section, in the Greek alphabet). In contrast to mathematician Kimberling, Bartlett motivated his finding on artistic considerations. They may be subject to criticism similar to the ‘golden ratio debunking’, but here we focus on showing that his chi ratio is interesting as a number as such, with pleasant geometric properties, just as the golden ratio. Moreover, Bartlett’s construction of proportional rectangles using perpendicular diagonals, which is at the basis of his chi ratio, has interesting references in architecture and in art.

Definition of chi

Christopher Bartlett denoted a ratio, of which he thought it was an often recurring proportion in art, by χ, as this letter follows ϕ in the Greek alphabet and because it seems to follow some of the pleasing mathematical properties of $\phi = (1+\sqrt{5})/2 = 1.618\ldots$ (see [2], [5], [6], [7], [9], [10]). One of them is that a rectangle of width 1 and length ϕ can be subdivided into a square and another rectangle proportional to the original rectangle. That smaller proportional rectangle will have width $1/\phi$ and length 1, and the square a side 1 because $\phi = 1/\phi + 1$ (see Figure 1). They can be easily constructed using a diagonal and the perpendicular from a vertex not on that diagonal. Of course, there are (many) other methods to get ϕ, but the latter will be applied here in other instances.

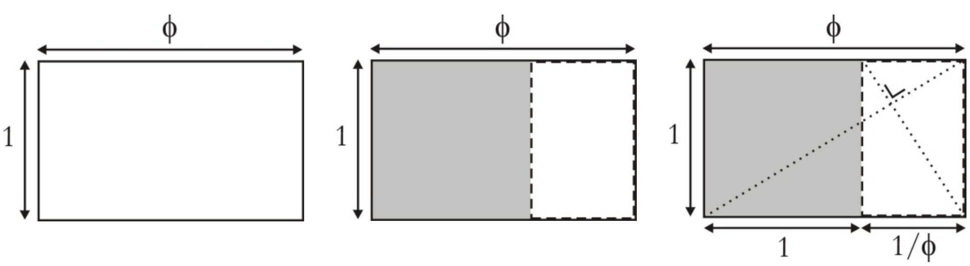


Figure 1: *A rectangle with proportion ϕ can be subdivided in a square (grey) and a rectangle proportional to the original rectangle (dashed line).*

Bartlett proposed to continue this procedure, that is, he wondered what kind of rectangle would be constructed in a similar way, if the starting point were a golden rectangle, instead of a square? He proposed to consider a rectangle with width 1 and length ϕ or one with width $1/\phi$ and length 1, in order to extend the ‘golden rectangle’ by a rectangle of the same proportions as the combination of it with a golden section rectangle. Again, the construction can be easily checked, using a diagonal and the perpendicular from a vertex not on that diagonal. In the case where the original rectangle has width 1 and length ϕ, Bartlett used the notation $1/\chi'$ for the width of the added rectangle (while it will have length 1). Thus, the combination of both will have width 1 and length χ'. In the case where the original rectangle has width $1/\phi$ and length 1, he denoted the width of the added rectangle by $1/\chi$ (again with length 1). Now, the combination of both will have width 1 and length χ (see Figure 2).

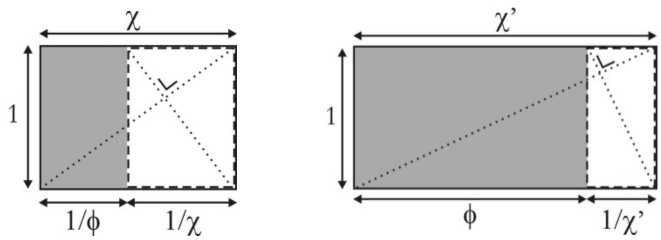


Figure 2: *Subdividing a rectangle with proportions χ or χ' in a golden rectangle (grey) and a rectangle proportional to the original rectangle (dashed line).*

The value of χ follows from the equation $\chi = 1/\phi + 1/\chi$, as it transforms in the quadratic equation $\chi^2 - (1/\phi)\chi - 1 = 0$. Thus, $\chi = \frac{(1/\phi)\pm\sqrt{(1/\phi)^2+4}}{2} = \frac{\sqrt{5}-1\pm\sqrt{22-2\sqrt{5}}}{4}$. The positive solution is 1.355… and this corresponds to what Bartlett called 'the chi ratio': χ =1.355….

Similarly, χ' follows from the equation $\chi' = \phi + 1/\chi'$, which transforms into $\chi'^2 - \phi\chi' - 1 = 0$, so that $\chi' = \frac{\phi\pm\sqrt{(1/\phi)^2+4}}{2} = \frac{1+\sqrt{5}\pm\sqrt{22+2\sqrt{5}}}{4}$. Now, the positive solution is 2.095…. and Bartlett denoted this by χ' = 2.095…

A referee of a first version of this paper suggested χ' could be called "the silver mean of the golden mean", since the golden and silver mean (and bronze, etc. …) refer to the positive roots $x = \frac{n+\sqrt{(n)^2+4}}{2}$ of $x^2 - nx - 1 = 0$, for n = 1, 2, … (and n=3, …). Similarly, $\chi = \frac{(1/\phi)\pm\sqrt{(1/\phi)^2+4}}{2}$ could be called "the silver mean of the inverse of golden mean".

Referee reports also pointed out a so-called 'meta-golden rectangle' had already been introduced in the 'On-line Encyclopedia of Integer Sequences' (see [15], series A188635). Indeed, three short paragraphs define it as a rectangle such that if a golden rectangle is removed from one end, the remaining rectangle is golden. Using an easy and pedagogically interesting technique (see [12]), Clark Kimberling obtained the series figuring in Sloane's encyclopedia. It is the continued fractions for χ': [2, 10, 2, 40, 10, 2, 2, 1, 14, 1, ...]. A similar expression can be proposed for χ: [1, 2, 1, 4, 3, 3, 1, 5, 1, 1, 2, 1, 36, 4, 9, …]. Thus,

$$\chi = 1 + \frac{1}{2+\frac{1}{1+\cdots}} \qquad \text{and} \qquad \chi' = 2 + \frac{1}{10+\frac{1}{2+\ldots}}$$

The label of 'meta-golden section' was adopted here, since Clark Kimberling's name for that proportion dates from 2011, and that was prior to Christopher Bartlett. The fact that mathematician Kimberling and artist Bartlett independently and almost simultaneously came up with an identical constant perhaps well illustrates the finding was inevitable and 'in the air'. Yet, the name 'meta-golden section' does not turn up many hits yet on the internet, nor has there already been any publication about it in a journal about mathematics or art. This is surprising, because the related golden section enjoys an enormous popularity in (pseudo-) science.

Note this was only the case since the last century, mainly due to the works of the German psychologist Adolf Zeising (1810-1876) and the Romanian diplomat Prince Matila Ghyka (1881 – 1965). They imagined the golden section was used for esthetic purposes as well, that is, that the proportion 1.618… would have been omnipresent in art and architecture. There are no historic or scientific arguments for it (see [13]), but perhaps the use of an approximation of Bartlett's chi ratio in art is not that odd. Dutch architect Dom Hans van der Laan's 'plastic number' $\psi = 1.324\ldots$ (1928; see [17]) and Spanish architect Rafael de la Hoz's 'Cordovan proportion' $c = 1.306\ldots$ (1973; see [11]) are both close to the classical proportion of 4/3, called the 'sesquitertia' (see [14]). Unquestionably, the 'sesquitertia' was used by Vitruvius, Pacioli and Leonardo, in contrast to the golden section. Moreover, since Bartlett (unconsciously!) followed the right angle construction that was so treasured by Le Corbusier, his discovery might resist a future 'chi ratio debunking' (see [1], [3], [4], [8]). Still, the goal of the present paper is the mathematical properties of χ, and so we will limit ourselves to those.

Some immediate properties

From

$$\chi^2 - (1/\phi)\chi - 1 = 0 \quad \text{and} \quad \chi'^2 - \phi\chi' - 1 = 0$$

follows that

$$\chi = \sqrt{1 + (1/\phi)\chi} \quad \text{and} \quad \chi' = \sqrt{1 + \phi\chi'}$$

Thus

$$\chi = \sqrt{1 + \phi^{-1}\sqrt{1 + \phi^{-1}\sqrt{1 + \ldots}}} \quad \text{and} \quad \chi' = \sqrt{1 + \phi\sqrt{1 + \phi\sqrt{1 + \cdots}}}$$

This can be compared to the well-known root expression for ϕ:

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}.$$

Also, from

$$\chi^2 - (1/\phi)\chi - 1 = 0 \quad \text{and} \quad \chi'^2 - \phi\chi' - 1 = 0,$$

it follows that

$$\chi = (1/\phi) + 1/\chi \quad \text{and} \quad \chi' = \phi + 1/\chi'$$

Thus

$$\chi = \left(\frac{1}{\phi}\right) + \frac{1}{\left(\frac{1}{\phi}\right) + \frac{1}{\left(\frac{1}{\phi}\right) + \cdots}} \quad \text{and} \quad \chi' = \phi + \frac{1}{\phi + \frac{1}{\phi + \ldots}}$$

This can be compared to the well-known continued fraction for ϕ:

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}}.$$

The geometric interpretations are similar too (see Figure 3).

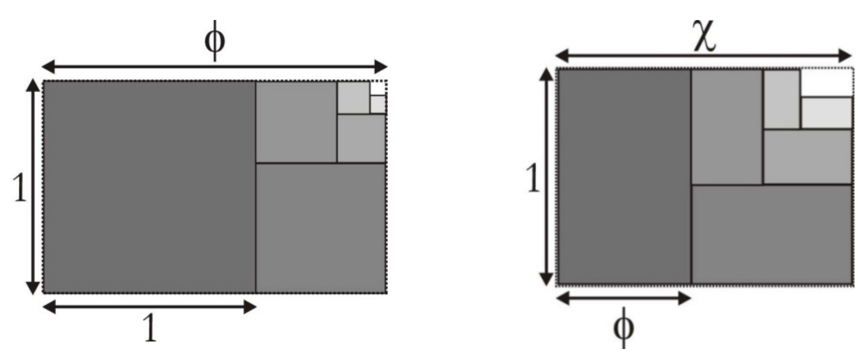


Figure 3: *Interpretations of the continued fractions for φ and for χ.*

The fractions 2, 3/2, 5/3, 8/5, … successively taken from the latter continued fraction give rise to the Fibonacci sequence (F_n): (1, 1,) 2, 3, 5, 8, … in which each term is the sum of the previous two terms. We follow a similar approach with a generalized continued fraction for χ' and consider:

$$\phi+1\ ,\ \phi+\frac{1}{\phi+1},\ \phi+\frac{1}{\phi+\frac{1}{\phi+1}},\ \phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+1}}},\ \phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+1}}}},\ \ldots$$

That is, taking into account that $\phi^2=\phi+1$:

$$\phi+1,$$
$$\phi+\frac{1}{\phi+1}=(\phi^2+\phi+1)/(\phi+1)=(2\phi+2)/(\phi+1)=2,$$
$$\phi+\frac{1}{\phi+\frac{1}{\phi+1}}=\phi+1/(2)=(2\phi+1)/2,$$
$$\phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+1}}}=\phi+2/(2\phi+1)=(2\phi^2+\phi+2)/(2\phi+1)=(3\phi+4)/(2\phi+1),$$
$$\phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+\frac{1}{\phi+1}}}}=\phi+(2\phi+1)/(3\phi+4)=((3\phi^2+4\phi)+(2\phi+1))/(3\phi+4)$$
$$=((3\phi+3+(6\phi+1))/(3\phi+4)=(9\phi+4)/(3\phi+4)$$

…

This gives rise to a series (H_n): (ϕ+1, 2,) 2ϕ+1, 3ϕ+ 4, 9ϕ+4, 16ϕ+13, 38ϕ+20, 74ϕ+51,… and from its construction it follows the series H_{n+1}/H_n converges to χ' = 2.095… Its rule is simple and somehow similar to the Fibonacci sequence: if aϕ+b, cϕ+d are consecutive terms, than the next one is (a+c+d)ϕ+(b+c) (Note that if the more obvious rule, (aϕ+b)+(cϕ+d) as in the Fibonacci series, is used, the series H_{n+1}/H_n would converge to ϕ). It is not an integer sequence, in contrast to the Fibonacci sequence. The first numbers in the series are: (2.618…, 2,) 4.236…, 8.854…, 18.562…, 38.888…, 81.484…, 170.732…, … .

As $\phi = 1/2 + \sqrt{5}/2$ and $\chi = \sqrt{5}/4 - 1/4 + (\sqrt{(22-2\sqrt{5})})/4$ are rational expressions with square roots, they can be constructed with compass and ruler in a finite number of steps (see Figure 4).

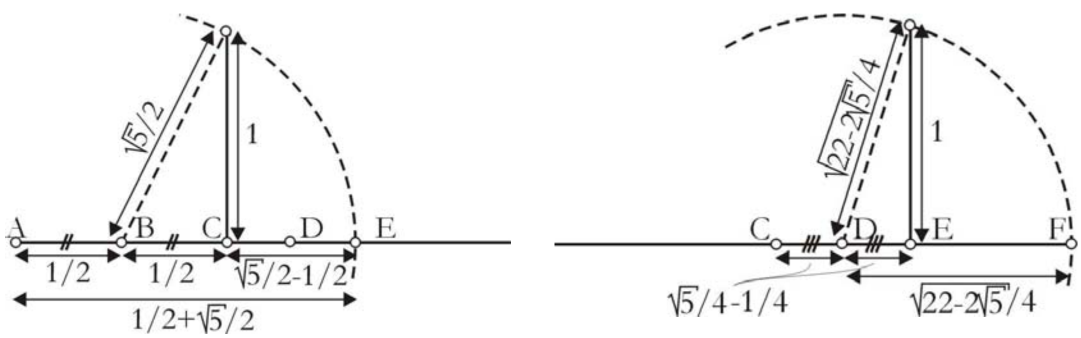


Figure 4: *Construction of the φ - and χ-ratio.*

From the above equations it follows that the solutions of $x^2 - (1/\phi)x - 1 = 0$ are χ and −1/χ, while the solutions of $x^2 + \phi x - 1 = 0$ are χ' and −1/χ' and thus they are the solutions of the following polynomial with integer coefficients:

$$(x-\chi)(x+\frac{1}{\chi})(x+\chi')\left(x-\frac{1}{\chi'}\right)=\left(x^2-(\chi-\frac{1}{\chi})\cdot x-1\right)\left(x^2+(\chi'-\frac{1}{\chi'})\cdot x-1\right)$$
$$=(x^2-\phi x-1).(x^2+(1/\phi)x-1)=x^4-(\phi-1/\phi)x^3-3x^2+(\phi-1/\phi)x+1=x^4-x^3-3x^2+x+1$$

This property, pleasing from a purely mathematical point of view, was noticed by A. Redondo (see [14]), while discussing this paper and the related notion of so-called 'minimal polynomial'. In the scope of the present paper it might lead too far to explain this here, but we wanted to mention this quartic $x^4 - x - 3x^2 + x + 1$ as it suggests the recurrence relation $s_{n+4} = s_{n+3} + 3s_{n+2} - s_{n+1} - s_n$. For $s_0 = s_1 = s_2 = 0$ and $s_3 = 1$, this gives rise to the sequence 0, 0, 0, 1, 1, 4, 6, 16, 29, 67, … (see [16]). The sequence turns out to be a Chebyshev related transform of the Fibonacci numbers, thus linking χ to the Fibonacci numbers and so again to ϕ.

Extending arbitrary rectangles proportionally

Bartlett's method with the orthogonal diagonals linking a square to a golden section rectangle, and a golden section rectangle to a chi ratio rectangle of course works equally well when starting with an arbitrary rectangle. In case $x < \phi$, the proportion of the longer side to the shorter side gives rise to the equation $1/(x - 1/x) = x/(x^2 - 1) = \rho$, if ρ is the given proportion. In case $x > \phi$, the proportion of the longer side to the shorter side gives rise to the equation $(x - 1/x)/1 = (x^2 - 1)/x = \rho$, if ρ is the given proportion (see Figure 5).

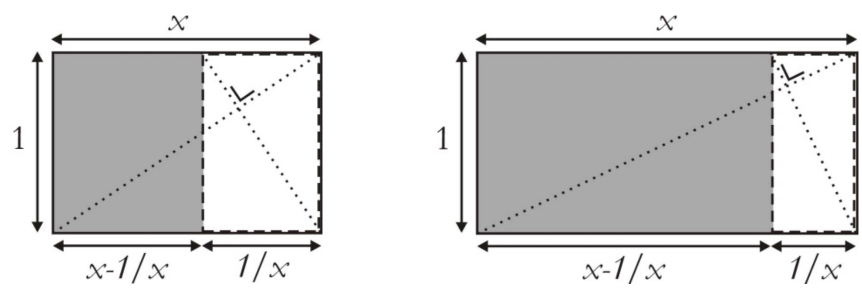


Figure 5: *Finding proportional rectangles in case $x < \phi$ (left) and $x > \phi$ (right).*

For instance, if one starts with a 1 by 2 rectangle (1 and 2 are the 2nd and 3th Fibonacci numbers), $\rho = 2/1$. Here, $x = 1+\sqrt{2}$ will provide a subdivision in a 1 by $(1+\sqrt{2})$ rectangle in a 1 by 2 rectangle and a 1 by $1/(1+\sqrt{2})$ rectangle (see Figure 6). The number $1+\sqrt{2}$ is called the 'silver mean'. If one starts with a 2 by 3 rectangle (2 and 3 are the 3th and 4th Fibonacci numbers), $\rho = 3/2$. Now an $x = 2$ will provide a subdivision in a 1 by 3/2 rectangle and a 1 by 1/2 rectangle.

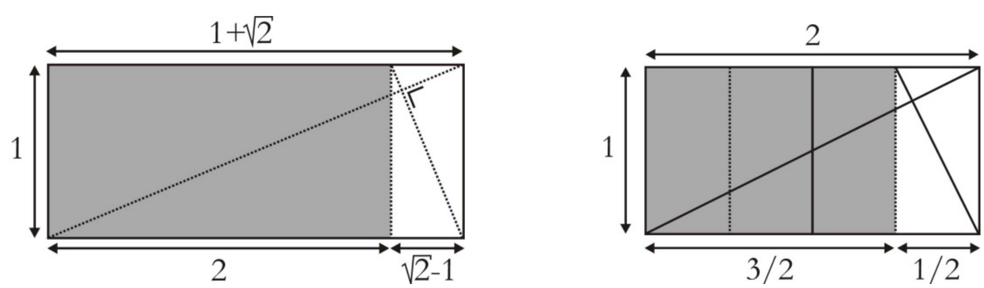
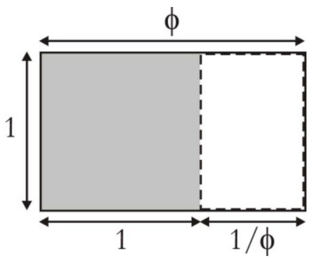
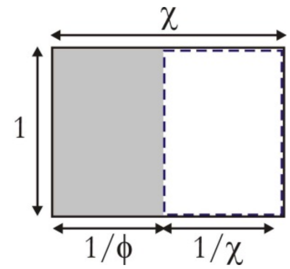


Figure 6: *Extending a 1 by 2 rectangle (left), and a 1 by 3/2 rectangle (right).*

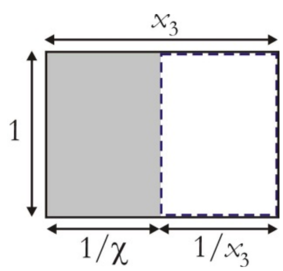
We can generalize Bartlett's method: what about extending a chi ratio rectangle with a rectangle proportional to that rectangle together with the chi ratio rectangle? It yields a rectangle with length 1.434… (and width 1). If this method is continued 'at infinity', the equation $x/(x^2 - 1) = x$ is obtained, and its positive solution is the square root of 2 rectangle, that is, the (European) DIN A4 paper format (see Figure 7). Of course, this is not so surprising, since the procedure $x_n/(x_n^2 - 1) = x_{n+1}$ yields $\sqrt{2}$ regardless if the initial value x_0 equals ϕ or not, but the construction method seemed neat enough to mention it.



A square extended to a ϕ = 1.618... golden rectangle.



A golden ϕ = 1.618... rectangle extended to a χ = 1.355... chi ratio rectangle.



A χ = 1.355... chi ratio rectangle extended to an x_3 = 1.434…rectangle.

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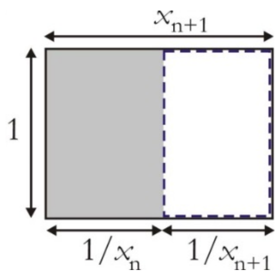

An x_n ratio rectangle extended to an x_{n+1} ratio rectangle.

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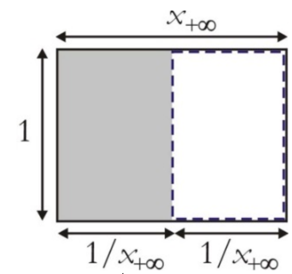

An $x_{+\infty}$ = √2 = 1.414… ratio rectangle extended to an $x_{+\infty}$ ratio rectangle.

Figure 7: *Extending Bartlett's method at infinitum.*

Paper folding

A well-known 'origami' procedure (see [18]) allows constructing a regular pentagon, and thus the golden section, using paper folding. A method for approximating the golden ratio can be obtained using Bartlett's diagonal construction that led to the chi ratio. First, we notice a quick method for creating proportional rectangles by folding, without any instrument or calculator. Say a sheet of paper has width 1 and an arbitrary length x. After folding one diagonal, the perpendicular through one vertex not on that diagonal can be obtained easily by folding the paper such that the folds correspond (see Figure 8). It yields a length $1/x$.

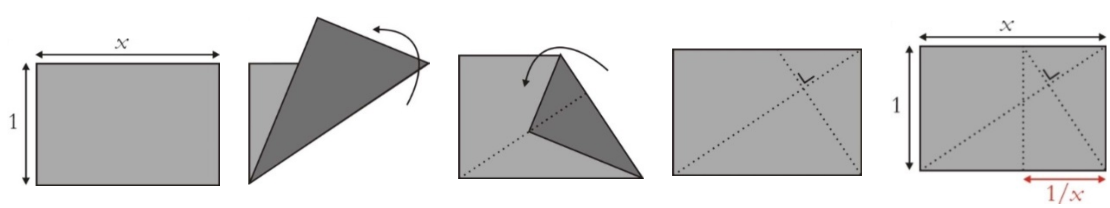

Figure 8: *Folding a proportional rectangle, given an arbitrary rectangle.*

As a square can be easily folded (see Figure 9), a sequence of squares gives rectangles of integer length (given the width is 1). Repetition of the folding method to a rectangle of width 1 and length n, gives a rectangle of width 1 and length n+1/(n+1/n), or as many steps as desired in the expansion of the continued fraction $\mathrm{n} + \cfrac{1}{\mathrm{n}+\cfrac{1}{\mathrm{n}+\cfrac{1}{\mathrm{n}+\dots}}}$. For instance, if n = 2, $2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}$ will correspond to the silver section 1 + √2, and thus approximations for that irrational number based on its continued fraction can be folded easily. The folding process based on n=3 given in the illustration will converge to the so-called bronze mean, (3 + √13)/2.

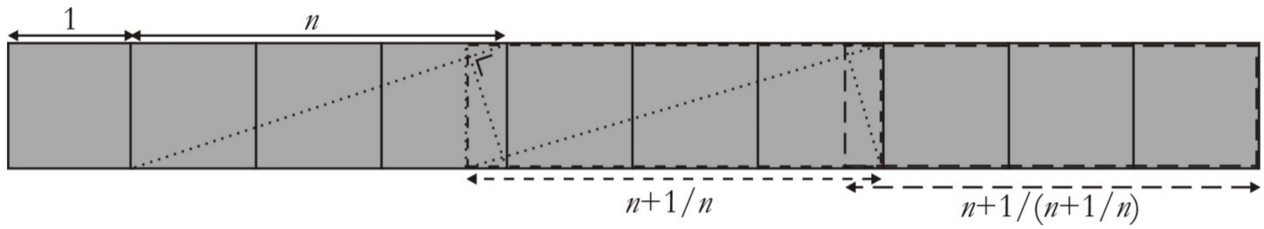

Figure 9: *Folding a square is easy, and thus so is folding a rectangle of any integer length n (here, n = 3); hence a continued fraction can be folded to a desired degree of precision.*

An 'application' is a paper folding construction of ($1/m$ + $1/n$) = (m+n)/mn given any two lengths m and n. This proportion is related to the harmonic mean $H = 2/(1/m + 1/n)$. The construction easily follows from the above (see Figure 10).

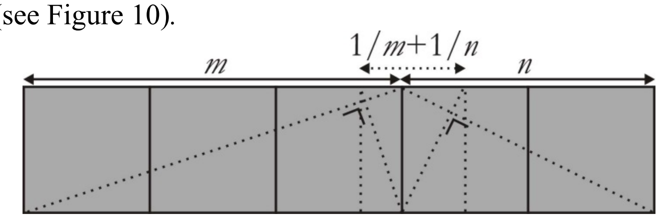

Figure 10: *Folding (m+n)/mn given m and n (in the figure, m=3 and n=2).*

The folding procedure works for a rectangle with an arbitrary length and repetition of the folding method starting from a rectangle with width 1 and arbitrary length x will, 'in the end', provide the golden section (see Figure 11).

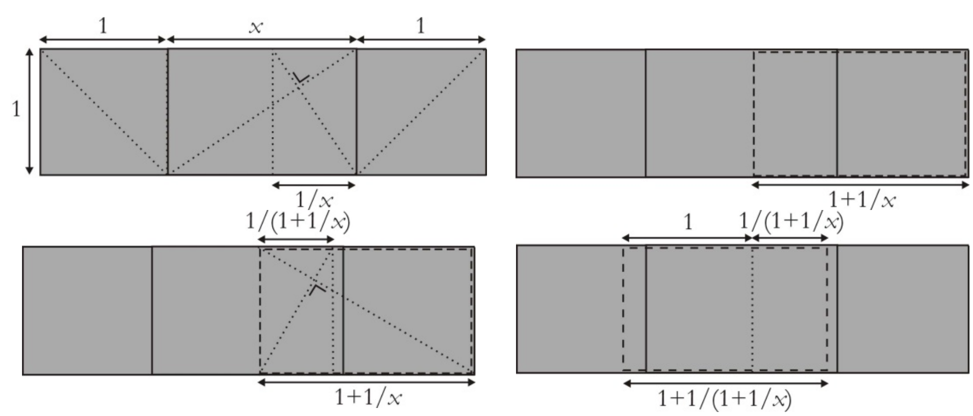

Figure 11: *Folding a rectangle with width x yields a rectangle with width 1+1/(1+(1/1… 1+/(1+x)…), an approximation for ϕ.*

Paintings incorporating φ and χ

The golden section debunking mentioned above has amply shown the esthetic interpretation of classical paintings using the golden section suffers from an excess of fantasy. Some modern artists however particularly appreciated an excess of fantasy, and deliberately based their composition on the mythical number. This was the case for Dali, in his 'The Sacrament of the Last Supper'. Similarly, Bartlett used the meta-golden section as a compositional inspiration for his painting 'Venice' (see Figure 12; three of his paintings figure in the Bridges art exhibition). In fact, it was this way Bartlett came to re-discover the proportion Kimberling had obtained based on a purely mathematical reasoning. And isn't remarkable that a mathematician and an artist independently came to the same idea of proposing the same constant?

 

Figure 12: *Paintings by Dali and Bartlett, based on, respectively, the golden and meta-golden section (left: tracing by the author; right: image used with permission from the painter).*

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